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The standard model and its generalizations in the Epstein–Glaser approach to renormalization theory: II. The fermion sector and the axial anomaly

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Abstract
We complete our study of non-Abelian gauge theories in the framework of the Epstein–Glaser approach to renormalization theory including in the model an arbitrary number of Dirac fermions. We consider the consistency of the model up to the third order of the perturbation theory. In the second order we obtain pure group theoretical relations expressing a representation property of the numerical coefficients appearing in the left- and right-handed components of the interaction, Lagrangian. In the third order of the perturbation theory we obtain the the condition of cancellation of the axial anomaly.

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1. Introduction

In some preceding papers [16,17] we have extended results of Aste *et al* [3,4,13] concerning the uniqueness of the non-Abelian gauge theory describing the consistent interaction of bosons of spin 1. It appeared that the gauge invariance principle is a natural consequence of the description of spin-one particles in a factor Hilbert space: gauge invariance expresses the possibility of factorizing the S-matrix to the physical space, which is usually constructed using the existence of a supercharge Q according to the cohomological-type formula $\mathcal{H}_{\text{phys}} = \text{Ker}(Q)/\text{Im}\,(Q)$. The obstructions to such a factorization process are the well known anomalies. The case when the spin-one bosons of non-null mass are admitted in the game was studied in [4, 13] for the concrete case of the electro-weak interaction i.e. when the gauge group is exactly $SU(2) \times U(1)$.

In [17] we analysed the same problem considering that the spin-one bosons can have non-null masses; we did not impose any restriction on their number and masses and we did not took into account the matter fields. Similar results were obtained in [24]. We have obtained, only from the condition of absence of the anomaly up to the second order, the existence of a Lie algebra $\mathfrak{g}$ and the existence of a representation of this Lie algebra pertaining to the Higgs fields.

In this paper, we consider the effect of including Dirac fermions. In this way we are able to investigate a truly realistic model of gauge interactions of elementary particles and, in particular, to see what are the restrictions on such a model determined by the cancellation of all anomalies. The main results are the following ones.

(A) The cancellation of the anomaly in the second order of the perturbation theory brings new relations on the numerical coefficients of the left- and right-handed components of the interaction Lagrangian. More precisely, new group theoretical properties appear:
   (i) The coefficients of the vectorial and pseudo-vectorial couplings can be organized as two representations of the gauge algebra, t_a^+ and t_a^- with $a, b, \ldots = 1, \ldots, r$ group indices; the usual notations are t_a^R and t_a^L.
   (ii) The coefficients of the scalar and pseudo-scalar couplings can be organized as some tensor operators.
   Some of these relations have been obtained from different considerations in [6,22,26].
   (iii) Some conditions on the couplings of the Higgs fields appear if one imposes the additional requirement that no finite renormalizations of degree greater than 4 are allowed. This condition gives the usual expression for the Higgs potential [4, 13] for the case of the standard model (SM).

(B) The cancellation of the anomaly in the third order of the perturbation theory gives, essentially, the usual condition of cancellation of the axial anomaly:

$$A_{abc} \equiv \mathrm{Tr}\left(t_a^+\{t_b^+, t_c^+\}\right) - \mathrm{Tr}\left(t_a^-\{t_b^-, t_c^-\}\right). \tag{1.0.1}$$

This is the expression of the Adler–Bardeen–Bell–Jackiw anomaly [1,2,5,7,20–22,25,27]. So, we obtain the usual condition of cancellation of the axial anomaly from the rigorous causal approach to renormalization theory.

The structure of the paper is the following. In the next section we define the model and construct the interaction Lagrangian including Dirac fermions. Then in section 3 we outline the general setting for the study of the renormalization theory, the general structure of Ward identities and some facts about distribution splitting. In section 4 we construct the S-matrix up to the second order of the perturbation theory. For the case without matter fields we reobtain the results of [17]. Then we consider the coupling of Yang–Mills fields with Dirac fermions and, as anticipated above, we obtain the group-theoretical information explained above. The complete analysis of these relation—we refer especially to (4.1.6)—is not available in the literature in full generality, at least to our knowledge; this subject deserves further investigation. We also analyse the conservation of the BRST current in the second order of the perturbation theory. In section 5 we go to the third order of the perturbation theory. We investigate the Dirac fermionic sector and we obtain the new conditions on the fermionic representations from above. Finally we particularize the formalism for the case of the SM with one generation of Dirac particles.

For the sake of clarity of the rather long and intricate analysis we adopt the mathematical *definition–theorem* style of presenting various assertions and computations.

2. General description of the vector Bosons

2.1. Massive Yang–Mills fields

In [16] and [17] we have started from the following two facts:

(1) a system of free zero-mass vector bosons can be described in a Hilbert space generated from the vacuum Ω by applying the free fields $A_\mu,\ u,\ \tilde{u}$ of zero mass and a factorization procedure induced by a supercharge operator;

(2) a system of free vector bosons of mass $m > 0$ can be described in a Hilbert space generated from the vacuum Ω by applying the free fields $A_\mu,\ u,\ \tilde{u},\ \Phi$ of mass m and a factorization procedure induced by a supercharge operator.

Here A_μ is a boson vector field, u and $\tilde{u}$ are scalar Fermi fields and Φ is a scalar boson field; the fields $u,\ \tilde{u}, \Phi$ are usually called ghost fields.

For the Yang–Mills model we somehow combine these two cases. We consider the auxiliary Hilbert space $\mathcal{H}^{gh,r}_{\mathrm{YM}}$ generated from the vacuum Ω by applying the free fields $A_{a\mu},\ u_a,\ \tilde{u}_a,\ \Phi_a\ a = 1, \ldots, r$, where the first one has vector transformation properties with respect to the Poincaré group and the others are scalars. In other words, every vector field has three scalar partners. Also $u_a,\ \tilde{u}_a\quad a = 1, \ldots, r$ are fermion and $A_\mu,\ \Phi_a\ a = 1, \ldots, r$ are boson fields.

We have two distinct possibilities for distinct indices a:

(I) Fields of type I correspond to an index a such that the vector field A^μ_a has non-zero mass m_a. In this case we suppose that all the other scalar partner fields $u_a,\ \tilde{u}_a,\ \Phi_a$ have the same mass m_a.

(II) Fields of type II correspond to an index a such that the vector field A^μ_a has zero mass. In this case we suppose that the scalar partner fields $u_a,\ \tilde{u}_a$ also have zero mass but the scalar field Φ_a can have a non-zero mass: $m^H_a \geqslant 0$. It is convenient to use the compact notation

$$m^*_a \equiv \begin{cases} m_a & \text{for}\quad m_a \neq 0 \\ m^H_a & \text{for}\quad m_a = 0. \end{cases} \tag{2.1.1}$$

Then the following equations of motion describe the preceding construction:

$$\begin{aligned} &(\Box + m_a^2)u_a(x) = 0 && (\Box + m_a^2)\tilde{u}_a(x) = 0 \\ &(\Box + (m^*_a)^2)\Phi_a(x) = 0 && a = 1, \ldots, r. \end{aligned} \tag{2.1.2}$$

We also postulate the following canonical (anti)commutation relations:

$$\begin{aligned} &[A_{a\mu}(x), A_{b\nu}(y)] = -\delta_{ab} g_{\mu\nu} D_{m_a}(x - y) \times \mathbf{1} \\ &\{u_a(x), \tilde{u}_b(y)\} = \delta_{ab} D_{m_a}(x - y) \times \mathbf{1} \\ &[\Phi_a(x), \Phi_b(y)] = \delta_{ab} D_{m^*_a}(x - y) \times \mathbf{1} \end{aligned} \tag{2.1.3}$$

(all other (anti)commutators are null).

In this Hilbert space we suppose, given a sesquilinear form $\langle\cdot,\cdot\rangle$ such that

$$\begin{aligned} &A_{a\mu}(x)^\dagger = A_{a\mu}(x) && u_a(x)^\dagger = u_a(x) \\ &\tilde{u}_a(x)^\dagger = -\tilde{u}_a(x) && \Phi_a(x)^\dagger = \Phi_a(x). \end{aligned} \tag{2.1.4}$$

The ghost degree is ± 1 for the fields $u_a,\ \tilde{u}_a, a = 1, \ldots, r$ and 0 for the other fields.

One can define the BRST *supercharge* Q by

$$\begin{aligned} &\{Q, u_a\} = 0 && \{Q, \tilde{u}_a\} = -\mathrm{i}(\partial_\mu A^\mu_a + m_a \Phi_a) \\ &[Q, A^\mu_a] = \mathrm{i}\partial^\mu u_a && [Q, \Phi_a] = \mathrm{i} m_a u_a \qquad \forall a = 1, \ldots, r \end{aligned} \tag{2.1.5}$$

and

$$Q\Omega = 0. \tag{2.1.6}$$

Then one can justify that the physical Hilbert space of the Yang–Mills system is a factor space

$$\mathcal{H}^{r}_{\rm YM} \equiv \mathcal{H} \equiv {\rm Ker}(Q)/{\rm Ran}(Q). \tag{2.1.7}$$

The sesquilinear form $\langle\cdot,\cdot\rangle$ induces a *bona fide* scalar product on the Hilbert factor space. The factorization process leads to the following physical particle content of this model:

- For $m_a > 0$ the fields $A^{\mu}_a,\ u_a,\ \tilde{u}_a,\ \Phi_a$ describe a particle of mass $m_a > 0$ and spin 1; these are the so-called *heavy bosons* [17].
- For $m_a = 0$ the fields $A^{\mu}_a,\ u_a,\ \tilde{u}_a$ describe a particle of mass 0 and helicity 1; the typical example is the *photon* [16].
- For $m_a = 0$ the fields Φ_a describe a scalar fields of mass m^H_a; these are the so-called *Higgs fields*.

This framework is sufficient for the study of the SM of the electro-weak interactions: indeed one takes $r = 4$ and considers that there are three fields of type I and one field of type II. The scalar field appearing in the last case can be considered as the Higgs field.

To also include quantum chromodynamics one must consider that there is a third case:

(III) Fields of type III correspond to an index a such that the vector field A^{μ}_a has zero mass and the scalar partners $u_a,\ \tilde{u}_a$ also have zero mass but the scalar field Φ_a is absent.

In [24] and [14] the model is constructed somewhat differently: one eliminates the fields of type II and includes a number of supplementary scalar bosonic fields φ_i of masses $m_i \geqslant 0$. In this framework one can consider for instance the very interesting Higgs–Kibble model in which there are no zero-mass particles, so the adiabatic limit probably exists.

One can preserve the general framework with only two types of index if we consider that in case II there are in fact three subcases (i.e. three types of index a for which $m_a = 0$):

(IIa) in this case $A_{a\mu},\ u_a,\ \tilde{u}_a,\ \Phi_a \not\equiv 0$;

(IIb) in this case $\Phi_a \equiv 0$;

(IIc) in this case $A_{a\mu},\ u_a,\ \tilde{u}_a \equiv 0$.

One must modify appropriately the canonical (anti-) commutation relations (2.1.3) to avoid contradiction for some values of the indices. One has some freedom of notation: for instance, one can eliminate case (IIa) if one includes the first three fields in case (IIb) and the last one in case (IIc). The relations (2.1.5) are not affected in this way.

Let us consider the set of Wick monomials $\mathcal{W}$ constructed from the free fields $A^{\mu}_a,\ u_a,\ \tilde{u}_a$ and Φ_a for all indices $a = 1, \ldots, r$; we define the BRST operator $d_Q : \mathcal{W} \to \mathcal{W}$ as the (graded) commutator with the supercharge operator Q. Then one can prove easily that

$$d_Q^2 = 0. \tag{2.1.8}$$

The class of observables on the factor space is defined as follows: an operator O: $\mathcal{H}^{gh,r}_{\rm YM} \to \mathcal{H}^{gh,r}_{\rm YM}$ induces a well defined operator $[O]$ on the factor space $\overline{{\rm Ker}(Q)/{\rm Im}\,(Q)} \simeq \mathcal{F}_m$ if and only if it verifies $d_Q O|_{{\rm Ker}(Q)} = 0$. Because of the relation (2.1.8) not all operators verifying the condition (2.1) are interesting. In fact, the operators of the type $d_Q O$ induce a null operator on the factor space; explicitly, we have

$$[d_Q O] = 0. \tag{2.1.9}$$

The canonical dimension $\omega(W)$ of a certain Wick monomial is defined according to the usual prescription. By definition, a *Wick polynomial* is a sum of Wick monomials.

We will construct a perturbation theory *á la* Epstein–Glaser using this set of free fields and imposing the usual axioms of causality, unitarity and relativistic invariance on the chronological products $T(x_1, \ldots, x_n)$. Moreover, we want the result to factorize to the physical Hilbert space in the adiabatic limit. This amounts to

$$\lim_{\epsilon \searrow 0} d_Q \int_{(\mathbb{R}^4)^{\times n}} \mathrm{d}x_1 \cdots \mathrm{d}x_n \, g_\epsilon(x_1) \cdots g_\epsilon(x_n) T(x_1, \ldots, x_n)|_{\mathrm{Ker}(Q)} = 0 \qquad \forall n \geqslant 1. \tag{2.1.10}$$

If this condition if fulfilled, then the chronological and the antichronological products do factorize to the physical Hilbert space and they give a perturbation theory verifying causality, unitarity and relativistic invariance.

One may raise at this point the rather serious objection that the adiabatic limit probably does not exist. One way to 'cure' this problem is to replace the condition of factorization (2.1.10) by the 'infinitesimal' version postulated in [3–13], namely

$$d_Q T(x_1, \ldots, x_n) = \mathrm{i} \sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} T_l^\mu(x_1, \ldots, x_n) \tag{2.1.11}$$

for some auxiliary chronological products $T_l^\mu(x_1, \ldots, x_n)$, $l = 1, \ldots, n$, which must be determined recurringly, together with the standard chronological products, and to construct the S-matrix $S(g)$ for a test function g, that is without performing the adiabatic limit $g \searrow 1$.

However, this point of view is not without problems. Indeed, if one imposes (2.1.11) instead of (2.1.10), then the S-matrix so constructed will *not* factorize to the physical space $\mathrm{Ker}(Q)/\mathrm{Im}\,(Q)$, which raises the question of its physical relevance. To this one must add the rather unpleasant fact that one abandons the consistency condition (2.1.10), which has a direct physical relevance: the possibility of constructing an S-matrix in the physical space $\mathrm{Ker}(Q)/\mathrm{Im}\,(Q)$) for an independent postulate (2.1.11). On the other hand, the rather close connection between (2.1.10) and (2.1.11) suggests that there must exist a common 'cure' for both types of problem. That is, if one can find a reasonable solution of the adiabatic limit problem, then it is reasonable to conjecture that one will be able to strengthen the mathematical status of (2.1.10) and, eventually, prove its equivalence with (2.1.11). In this case the consistency condition can be also written in the following form:

$$d_Q \int_{(\mathbb{R}^4)^{\times n}} \mathrm{d}x_1 \cdots \mathrm{d}x_n \, g_\epsilon(x_1) \cdots g_\epsilon(x_n) T(x_1, \ldots, x_n)|_{\mathrm{Ker}(Q)} = \mathcal{O}(\epsilon) \qquad \forall n \geqslant 1 \tag{2.1.12}$$

in the sense of the infinitesimal calculus of Dieudonné. In what follows, the interpretation of the right-hand side of the preceding relations will be 'an integrated divergence'. In other words, to avoid various problems we will use in fact the *formal adiabatic limit condition* given by (2.1.11). A more detailed discussion on this point can be found in [17].

By a *trivial Lagrangian* we mean a Wick expression of the type

$$L(x) = d_Q N(x) + \mathrm{i} \frac{\partial}{\partial x^\mu} L^\mu(x) \tag{2.1.13}$$

with $L(x)$ and $L^\mu(x)$ some Wick polynomials. The first term in the previous formula gives zero by factorization to the physical Hilbert space (according to a previous discussion) and the second one also gives zero in the adiabatic limit; this justifies the elimination of such an expression from the first-order chronological product $T(x)$.

If one completely exploits the condition of gauge invariance in the first order of perturbation theory, obtaining the generic form of the Yang–Mills interaction of spin-one bosons up to a trivial Lagrangian. We assume the summation convention of the dummy indices $a, b, \ldots = 1, \ldots, r$. The result from [17] is:

Theorem 2.1. *Let us consider the operator $T(x)$ defined on $\mathcal{H}_{\mathrm{YM}}^{gh,r}$ as a Lorentz-invariant Wick polynomial in $A_a^\mu(x)$, $u_a(x)$, $\tilde{u}_a(x)$, $\Phi_a(x)$ such that every term has canonical dimension* *three or four. If it verifies the formal adiabatic limit condition then it has, up to a trivial Lagrangian, the following form:*

$$\begin{aligned}T^{\mathrm{YM}}(x) = f_{abc}[&: A_{a\mu}(x)A_{b\nu}(x)\partial^{\nu}A^{\mu}_{c}(x) : - : A^{\mu}_{a}(x)u_{b}(x)\partial_{\mu}\tilde{u}_{c}(x) :]\\ &+f'_{abc}[: \Phi_{a}(x)\partial_{\mu}\Phi_{b}(x)A^{\mu}_{c}(x) : -m_{b} : \Phi_{a}(x)A_{b\mu}(x)A^{\mu}_{c}(x) :\\ &+m_{b} : \Phi_{a}(x)\tilde{u}_{b}(x)u_{c}(x) :]\\ &+f''_{abc} : \Phi_{a}(x)\Phi_{b}(x)\Phi_{c}(x) : +g_{abcd} : \Phi_{a}(x)\Phi_{b}(x)\Phi_{c}(x)\Phi_{d}(x) : .\end{aligned} \tag{2.1.14}$$

The various constants from the preceding expression are constrained by the following conditions:

- *the expressions f_{abc} are completely antisymmetric*

$$f_{abc} = -f_{bac} = -f_{acb} \tag{2.1.15}$$

  and verify

$$(m_a - m_b) f_{abc} = 0 \qquad \textit{iff} \quad m_c = 0 \quad \forall a, b = 1, \ldots, r \tag{2.1.16}$$

- *the expressions f'_{abc} are antisymmetric in the indices a and b:*

$$f'_{abc} = -f'_{bac} \tag{2.1.17}$$

  and verify the relation

$$(m^H_a - m^H_b) f'_{abc} = 0 \qquad \textit{iff} \quad m_a = m_b = m_c = 0 \quad \forall a, b = 1, \ldots, r \tag{2.1.18}$$

  and are connected to f_{abc} by

$$f_{abc}m_c = f'_{cab}m_a - f'_{cba}m_b \qquad \forall a, b, c = 1, \ldots, r \tag{2.1.19}$$

- *the expressions f''_{abc} remain undetermined for $m_a = m_b = m_c = 0$ and for the opposite case are given by*

$$f''_{abc} = \frac{1}{6m_c} f'_{abc}\left[(m^*_a)^2 - (m^*_b)^2 - m^2_a + m^2_b\right] \tag{2.1.20}$$

  for $m_c \neq 0$;
- *the expressions g_{abcd} are non-zero only for $m_a = m_b = m_c = m_d = 0$ and in this case they are completely symmetric.*

Remark 2.2. The presence of indices of type (IIb) and (IIc) is taken into account by requiring that the constants from $T(x)$ are null if one of the indices a, b, c takes such values. One can see that this does not affect the equations from the statement of the theorem.

We also have:

Corollary 2.3. *In the condition of the preceding theorem, one has*

$$d_Q T(x) = \mathrm{i}\partial_\mu T^\mu(x) \tag{2.1.21}$$

where

$$T^\mu = f_{abc}(: u_a A_{b\nu} F^{\nu\mu}_c : -\tfrac{1}{2} : u_a u_b \partial^\mu \tilde{u}_c :) + f'_{abc}(m_a : A^\mu_a \Phi_b u_c : + : \Phi_a \partial^\mu \Phi_b u_c :). \tag{2.1.22}$$

The expression $T(x)$ from the preceding theorem verifies the unitarity condition

$$T(x)^\dagger = T(x)$$

if and only if the constants f_{abc}, f'_{abc} and f''_{abc} have real values; it also verifies the causality condition

$$[T(x), T(y)] = 0 \qquad \forall x, y \in \mathbb{R}^4 \quad \text{s.t.} \quad (x-y)^2 < 0.$$

We close this subsection with some remarks.

Remark 2.4. One can see that the necessity of using ghost fields stems from the fact that it seems to be impossible to construct the interaction Lagrangian without them. However, from a fundamental point of view, one can consider them only as some catalysts [14] and hope that one will be able to reformulate the whole theory without them.

Remark 2.5. In the first-order analysis one can also use instead of the formal adiabatic limit condition (2.1.11) the more physical condition (2.1.10) because no problems connected with the adiabatic limit exist in this case. However, as noted in [13], the condition does essentially eliminate the tri-linear terms and one loses much of the information of the preceding theorem. This is another indication that one should work with the formal adiabatic limit condition.

Remark 2.6. In [8] one can find a discussion showing that trivial Lagrangians do not produce effects in the higher orders of perturbation theory.

2.2. Yang–Mills fields coupled to matter

We study here the possibility of coupling Yang–Mills fields to 'matter'. We suppose that we are given the Hilbert space of 'matter' $\mathcal{H}_{\text{matter}}$, which is usually also a Fock space. Then the coupled system is described in the tensor product Hilbert space $\mathcal{F}_{\text{YM}} \otimes \mathcal{H}_{\text{matter}}$. One can describe this Fock space considering $\tilde{\mathcal{H}}^{gh,r}_{\text{YM}} \equiv \mathcal{H}^{gh,r}_{\text{YM}} \otimes \mathcal{H}_{\text{matter}}$ with the corresponding supercharge operator and forming the quotient $\text{Ker}(Q)/\text{Im}\,(Q)$. We will consider here that the 'matter' is formed from Dirac fermions only.

First, we generalize theorem 2.1:

Theorem 2.7. *Let us consider the operator $T(x)$ defined on $\tilde{\mathcal{H}}^{gh,r}_{\text{YM}}$, which is a Lorentz-invariant Wick polynomial in $A^\mu_a(x)$, $u_a(x)$, $\tilde{u}_a(x)$, $\Phi_a(x)$ and the matter fields such that every term has canonical dimension three or four. Then $T(x)$ verifies the formal adiabatic limit condition if and only if, up to a trivial Lagrangian, it has the following form:*

$$T(x) = T^{\text{YM}}(x) + A^\mu_a(x) j_{a\mu}(x) + \sum_{m_a \neq 0} \frac{1}{m_a} \Phi_a(x) \partial_\mu j^\mu_a(x) + \sum_{m_a = 0} \Phi_a(x) j_a(x) + T_{\text{matter}}(x). \tag{2.2.1}$$

Here $T^{\text{YM}}(x)$ has been defined in theorem 2.1, $j_{a\mu}$ and j_a are Lorentz covariant currents built only from the matter fields with $\omega(j_{a\mu}) = 1, 2, 3$ and $T_{\text{matter}}(x)$ contains only the matter fields. Moreover the following conservation law should be valid:

$$\partial_\mu j^\mu_a(x) = 0 \qquad \forall m_a = 0. \tag{2.2.2}$$

The expression for $T(x)$ verifies the unitarity requirement if and only if we have

$$j^\mu_a(x)^\dagger = j^\mu_a(x) \quad \forall a = 1, \ldots, r \qquad j_a(x)^\dagger = j_a(x) \quad \forall m_a = 0 \tag{2.2.3}$$

and verifies the causality condition if and only if

$$[j^\mu_a(x), j^\nu_b(y)] = 0 \qquad (x-y)^2 < 0 \quad \forall a, b = 1, \ldots, r \tag{2.2.4}$$

$$[j_a(x), j_b(y)] = 0 \qquad (x-y)^2 < 0 \quad \forall m_a = m_b = 0 \tag{2.2.5}$$

$$[j^\mu_a(x), j_b(x)] = 0 \qquad (x-y)^2 < 0 \quad \forall m_b = 0. \tag{2.2.6}$$

Proof. Beside the terms considered in theorem 2.1 we have to include terms containing explicitly the Dirac fermions. Lorentz covariance and power counting limit these terms to $T_{\text{matter}}(x)$ and

$$T_{\text{matter}}(x) \equiv A^\mu_a(x) j_{a\mu}(x) + \Phi_a(x) j_a(x) \tag{2.2.7}$$

with $j_{a\mu}$ (j_a) a Lorentz covariant (invariant) operator. Proceeding in the same way as for the proof of theorem 2.1, we obtain a supplementary restriction, namely

$$m_a j_a = \partial_\mu j_a^\mu \qquad \forall a = 1, \ldots, r. \tag{2.2.8}$$

In other words, for $m_a = 0$ we obtain (2.2.2) and for $m_a \neq 0$ we obtain

$$j_a = \frac{1}{m_a} \partial_\mu j_a^\mu. \tag{2.2.9}$$

The expression from the statement emerges. The other assertions are straightforward, although rather tedious to verify. □

It is clear that if the Hilbert space of the matter fields is also a Fock space and the currents are build from Wick monomials, then the commutation relations (2.2.6) are always verified.

Corollary 2.8. *The following formula is true:*

$$d_Q T(x) = \mathrm{i} \frac{\partial}{\partial x^\mu} T^\mu(x) \tag{2.2.10}$$

where $T^\mu(x)$ is obtained by adding to the corresponding expression from the pure Yang–Mills case—see (2.1.22)—the following contribution due to the presence of matter:

$$T^\mu_{\text{matter}}(x) \equiv u_a(x) j_a^\mu(x). \tag{2.2.11}$$

Now we obtain in detail the structure of the interaction Lagrangian in the following two propositions. We have:

Proposition 2.9. *Suppose that the Dirac fermions generating $\mathcal{H}_{\text{matter}}$ are ψ_A of masses $M_A \geqslant 0$, $A = 1, \ldots, N$. Then the generic forms of the currents from the preceding theorem are*

$$j_a^\mu(x) =: \overline{\psi_A}(x)(t_a)_{AB}\gamma^\mu \psi_B(x) : + : \overline{\psi_A}(x)(t'_a)_{AB}\gamma^\mu\gamma_5\psi_B(x) : \tag{2.2.12}$$

and

$$j_a(x) =: \overline{\psi_A}(x)(s_a)_{AB}\psi_B(x) : + : \overline{\psi_A}(x)(s'_a)_{AB}\gamma_5\psi_B(x) : . \tag{2.2.13}$$

The causality conditions from theorem 2.7 are fulfilled and the hermiticity conditions are equivalent to the fact that the complex $N \times N$ matrices t_a, t'_a, s_a, $a = 1, \ldots, r$ are Hermitian and s'_a, $a = 1, \ldots, r$ anti-Hermitian.

The contributions with (without) the matrix γ_5 is called *axial* (*vectorial*) current. Let us define the *mass matrix* by

$$M_{AB} \equiv \delta_{A,B} M_A \qquad \forall A, B = 1, \ldots, N. \tag{2.2.14}$$

Then we have:

Proposition 2.10. *The following mass relations are true:*

$$s_a = \frac{\mathrm{i}}{m_a}[M, t_a] \qquad s'_a = \frac{\mathrm{i}}{m_a}\{M, t'_a\} \qquad \forall m_a \neq 0 \tag{2.2.15}$$

$$[M, t_a] = 0 \qquad \{M, t'_a\} = 0 \qquad \forall m_a = 0. \tag{2.2.16}$$

In particular, the matrices t_a, $\forall m_a = 0$, can be exhibited in a block diagonal structure (eventually after a relabelling of the Dirac fields) and the masses corresponding to the same block must be equal.

Proof. It is easy to show that the conservation law (2.2.8) is equivalent to the two relations from the statement. □

Corollary 2.11. *Let us define*

$$t_a^\epsilon \equiv t_a + \epsilon t_a' \qquad s_a^\epsilon \equiv s_a + \epsilon s_a' \qquad \forall a = 1, \ldots, r \tag{2.2.17}$$

where $\epsilon = \pm$. Then, the relations (2.2.15) and (2.2.16) are equivalent to

$$s_a^\epsilon = \frac{\mathrm{i}}{m_a}(M t_a^\epsilon - t_a^{-\epsilon} M) \qquad \forall m_a \neq 0 \tag{2.2.18}$$

$$M t_a^\epsilon = t_a^{-\epsilon} M \qquad \forall m_a = 0 \tag{2.2.19}$$

and the hermiticity conditions are equivalent to

$$(t_a^\epsilon)^* = t_a^\epsilon \qquad (s_a^\epsilon)^* = s_a^{-\epsilon} \qquad \forall a = 1, \ldots, r \quad \epsilon = \pm. \tag{2.2.20}$$

3. Perturbation theory

3.1. The general framework

We give here the basic ideas of a *multi-Lagrangian* perturbation theory following [15] and [18]. One can argue that the S-matrix is a formal series of operator valued distributions:

$$S(\boldsymbol{g}) = 1 + \sum_{n=1}^{\infty} \frac{\mathrm{i}^n}{n!} \int_{\mathbb{R}^{4n}} \mathrm{d}x_1 \cdots \mathrm{d}x_n\, T_{j_1,\ldots,j_n}(x_1, \ldots, x_n) g_{j_1}(x_1) \cdots g_{j_n}(x_n) \tag{3.1.1}$$

where $\boldsymbol{g} = \left(g_j(x)\right)_{j=1,\ldots,P}$ is a multi-valued tempered test function in the Minkowski space $\mathbb{R}^4$ that switches the interaction and $T_{j_1,\ldots,j_n}(x_1, \ldots, x_n)$ are operator-valued distributions acting in the Fock space of some collection of free fields with a common dense domain of definition D_0. These operator-valued distributions are called *chronological products* and verify some properties called *Bogoliubov axioms*. We note that there is a canonical association of the point x_i and the index j_i. One starts from a set of *interaction Lagrangians* $T_j(x)$, $j = 1, \ldots, P$ and tries to construct the whole series $T_{j_1,\ldots,j_n}$, $n \geqslant 2$.

We outline briefly the set of axioms imposed on the chronological products $T_{j_1,\ldots,j_n}$; we do not give the explicit formulæ because they are well known in the literature and can be found in the references quoted above.

- *Symmetry.* This axiom describes the behaviour of the chronological products with respect to the permutation of the couples (x_i, j_i).
- *Poincaré invariance.* This axiom describes the behaviour of the chronological products with respect to the action of the Poincaré group in the Fock space of the system. Essentially it is a tensorial covariance condition.
- *Causality.* This describes factorization properties of the chronological products for causally separated arguments. This seems to be the central axiom of this axiomatic approach; it plays a major rôle in other axiomatic schemes as well.
- *Unitarity.* This axiom is considered in the sense of formal series.

A *renormalization theory* is the possibility to construct such an S-matrix starting from the first-order terms: $T_j(x)$, $j = 1, \ldots, P$, which are linearly independent Wick polynomials called *interaction Lagrangians*, which should verify the corresponding axioms expressing the behaviour with respect to Poincaré transformations, Hermitian conjugation and commutation properties for spacelike separated arguments.

The case of a single Lagrangian corresponds to a single coupling constant, that is $P = 1$ and in that case the chronological products will be operators $T(X)$ without any indices. However, it is more convenient to consider that the interaction Lagrangian is given by the sum

$$T(x) = \sum c_j T_j(x) \tag{3.1.2}$$

with c_j some real constants. In this case, the chronological products of the theory are

$$T(X) = \sum c_{j_1} \ldots c_{j_n} T_{j_1,\ldots,j_n}(X). \tag{3.1.3}$$

It can be shown that that one must consider the given interaction Lagrangians $T_j(x)$ to be all Wick monomials of canonical dimension $\omega_j \leqslant 4$ $(j = 1, \ldots, P)$ acting in the Fock space of the system. Because the Fock space is generated by some free relativistic fields acting on the vacuum Ω it is easy to see that there are always covariance properties with respect to Poincaré transformations.

If there are non-Hermitian free fields acting in the Fock space, we have in general

$$T_j(x)^\dagger = T_{j^*}(x) \tag{3.1.4}$$

where $j \to j^*$ is a bijective map of the numbers $1, 2, \ldots, P$.

If there are Fermi or ghost fields acting in the Fock space, the causality property is in general

$$T_{j_1}(x_1)T_{j_2}(x_2) = (-1)^{\sigma_{j_1}\sigma_{j_2}} T_{j_2}(x_2)T_{j_1}(x_1) \qquad \forall x_1 \sim x_2. \tag{3.1.5}$$

Here σ_i is the number of Fermi and ghost field factors in the Wick monomial T_j; if σ_j is even (odd) we call the index j *even* (*odd*). One has to keep track of these signs in the symmetry axiom for the chronological products.

It is convenient to also let the index j have the value zero and we put by definition

$$T_0 \equiv \mathbf{1}. \tag{3.1.6}$$

Moreover, we define a new sum operation of two indices $j_1, j_2 = 1, \ldots, P$; this summation is denoted by + but should not be confused with the ordinary sum. By definition we have

$$T_{j_1+j_2}(x) = c : T_{j_1}(x)T_{j_2}(x) : \tag{3.1.7}$$

for some positive constant c. We define componentwise the summation for n-tuples $J = \{j_1, \ldots, j_n\}$. The new summation is non-commutative if Fermi or ghost fields are present.

We will use the notation

$$\omega_J \equiv \sum_{j\in J} \omega_j \tag{3.1.8}$$

and we call it the *canonical dimension* of $T_J(X)$.

Let us denote by $\omega(d)$ the order of singularity of the numerical distribution d. We use the definition from [23] although one can also use the scaling degree introduced by Steinmann (see [10]).

Then we add a new axiom, namely the following *Wick expansion property* of the chronological products is valid:

$$T_J(X) = \sum_{K+L=J} \epsilon t_K(X) W_L(X) \tag{3.1.9}$$

where (a) $t_K(X)$ are numerical distributions (the *renormalized Feynman amplitudes*), (b) the degree of singularity is restricted by the relation

$$\omega(t_K) \leqslant \omega_K - 4(n-1) \tag{3.1.10}$$

(c) ϵ is the sign originating from permutation of Fermi fields and (d) we have introduced the notation

$$W_J(X) \equiv : T_{j_1}(x_1) \cdots T_{j_n}(x_n) : . \tag{3.1.11}$$

Let us notice that from (3.1.9) we have

$$t_J(X) = \langle \Omega, T_J(X)\Omega \rangle. \tag{3.1.12}$$

In particular, these numerical distributions are Poincaré covariant; translation invariance implies that they are in fact distributions in $m = 4(|X| - 1)$ variables.

The recursive construction assumes that we have the expressions $T_J(X)$ for $|X| \leqslant n-1$ verifying all the properties and tries to construct them for $X = n$. The basic object is the commutator function:

$$D_{j_1,\ldots,j_n}(x_1,\ldots,x_{n-1};x_n) \equiv A'_{j_1,\ldots,j_n}(x_1,\ldots,x_{n-1};x_n) - R'_{j_1,\ldots,j_n}(x_1,\ldots,x_{n-1};x_n) \tag{3.1.13}$$

where

$$A'_{j_1,\ldots,j_n}(x_1,\ldots,x_{n-1};x_n) \equiv \sum\nolimits'_{X_1,X_2\in \mathrm{Part}(X)} (-1)^{|X_2|} T_{J_1}(X_1)\bar{T}_{J_2}(X_2) \tag{3.1.14}$$

and

$$R'_{j_1,\ldots,j_n}(x_1,\ldots,x_{n-1};x_n) \equiv \sum\nolimits'_{X_1,X_2\in \mathrm{Part}(X)} (-1)^{|X_2|} \bar{T}_{J_2}(X_2)T_{J_1}(X_1) \tag{3.1.15}$$

and the sums $\sum'$ run over the partitions verifying $X_2 \neq \emptyset$, $x_n \in X_1$.

The commutator function can be proved to be Poincaré covariant and to have causal support i.e. $\mathrm{supp}(D_{j_1,\ldots,j_n}(x_1,\ldots,x_{n-1};x_n)) \subset \Gamma^+(x_n) \cup \Gamma^-(x_n)$ where we use standard notations:

$$\Gamma^{\pm}(x_n) \equiv \{(x_1,\ldots,x_n) \in (\mathbb{R}^4)^n | x_i - x_n \in V^{\pm}\ \forall i = 1,\ldots,n-1\}. \tag{3.1.16}$$

Moreover, a formula similar to (3.1.9) is true:

$$D_J(X) = \sum_{K+L=J} \epsilon d_K(X) W_L(X) \tag{3.1.17}$$

where $d_K(X)$ are numerical distributions; in analogy to (3.1.12) we have

$$d_J(X) = \langle \Omega, D_J(X)\Omega \rangle. \tag{3.1.18}$$

It follows that the numerical distributions $d_J(X)$ have causal support i.e. $\mathrm{supp}(d_J(X)) \subset \Gamma^+(x_n)\cup\Gamma^-(x_n)$ and are $SL(2,\mathbb{C})$-invariant. Moreover, their degree of singularity is restricted by

$$\omega(d_K) \leqslant \omega_K - 4(n-1) \tag{3.1.19}$$

(this is the content of the *power counting theorem*). One knows that there exists a causal splitting

$$d_J = a_J - r_J \qquad \mathrm{supp}(a_J) \subset \Gamma^+(x_n) \qquad \mathrm{supp}(r_J) \subset \Gamma^-(x_n) \tag{3.1.20}$$

which is also $SL(2,\mathbb{C})$-invariant and such that the order of the singularity is preserved. So, there exists a $SL(2,\mathbb{C})$-covariant causal splitting:

$$D_J(X) = A_J(X) - R_J(X) \qquad |X| = n \tag{3.1.21}$$

with $\mathrm{supp}(A_{j_1,\ldots,j_n}(x_1,\ldots,x_{n-1};x_n)) \subset \Gamma^+(x_n)$ and $\mathrm{supp}(R_{j_1,\ldots,j_n}(x_1,\ldots,x_{n-1};x_n)) \subset \Gamma^-(x_n)$.

Let us define

$$T_J(X) \equiv A_J(X) - A'_J(X) = R_J(X) - R'_J(X). \tag{3.1.22}$$

Then these expressions satisfy the $SL(2,\mathbb{C})$-covariance, and causality axioms. One can also fix unitarity and symmetry.

We end this subsection with an important remark. Let us consider some general Wick polynomials

$$A_i(x) = \sum_j c_{ij} T_j(x) \qquad i = 1,2,\ldots. \tag{3.1.23}$$

Then we can define the chronological products

$$T(A_1(x_1),\ldots,A_n(x_n)) \equiv \sum_J c_{i_1 j_1}\cdots c_{i_n j_n} T_{j_1,\ldots,j_n}(x_1,\ldots,x_n). \tag{3.1.24}$$

One can find in [10] a system of axioms for the expressions $T(A_1(x_1),\ldots,A_n(x_n))$ which is equivalent to the Bogoliubov set of axioms.

3.2. Ward identities

As we said in section 2.1 the problem is to construct the whole series $T(X)$ such that one has the gauge invariance condition in all orders of the perturbation theory at the same time as the other Bogoliubov axioms.

In general we have something more general than relation (3.1.2)

$$T(x) = \sum c_j T_j(x) \qquad T^\mu(x) = \sum c_j^\mu T_j(x) \tag{3.2.1}$$

with c_j and c_j^μ some real constants; then we will have something more general than (3.1.3):

$$T(X) = \sum c_{j_1} \cdots c_{j_n} T_{j_1,\ldots,j_n}(X) \qquad T_l^\mu(X) = \sum c_{j_1} \ldots c_{j_l}^\mu \ldots c_{j_n} T_{j_1,\ldots,j_n}(X). \tag{3.2.2}$$

In particular, the following conventions hold:

$$T(\emptyset) \equiv \mathbf{1} \qquad T_l^\mu(\emptyset) \equiv 0 \qquad T_l^\mu(X) \equiv 0 \qquad \text{for} \quad x_l \notin X. \tag{3.2.3}$$

Then the gauge invariance condition (2.1.11) can be written more compactly as follows:

$$d_Q T(X) = \mathrm{i} \sum \frac{\partial}{\partial x_l^\mu} T_l^\mu(X). \tag{3.2.4}$$

We suppose that these relations are true up to order $|X| \leqslant n-1$ and investigate the possible obstructions in order n. The procedure used in [11, 12] and [16, 17] amounts to the following. Let us define the operator distributions $D(X)$ and $D_l^\mu(X)$ in analogy to the relations (3.2.2). Then it can be proved that we have

$$d_Q D(X) = \mathrm{i} \sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} D_l^\mu(X) \qquad |X| = n. \tag{3.2.5}$$

We can express this condition in terms of numerical distributions. According to the relation (3.1.9) and the Wick theorem we must have Wick expansions for the two expressions appearing in the preceding equation:

$$D(X) = \sum_J d_J(X) W_J(X) \qquad D_l^\mu(X) = \sum_J d_{l;J}^\mu(X) W_J(X). \tag{3.2.6}$$

The numerical distributions appearing in these relations have the following properties: they are Poincaré covariant, they have causal support and the order of singularity can be restricted according to the *power counting* formula:

$$\omega(d_J) + \omega_J \leqslant 4 \qquad \omega(d_{l;J}^\mu) + \omega_J \leqslant 4 \tag{3.2.7}$$

according to the power counting theorem.

One can rewrite (3.2.6) as follows:

$$D(X) = \sum_i d_i(X) W_i(X) \qquad D_l^\mu(X) = \sum_i d_i(X)\, W_{l;i}^\mu(X) + \sum_i d_i^\mu(X) W_{l;i}(X) \tag{3.2.8}$$

where d_i and d_i^μ can be taken to be linear independent over the vector space of smooth functions with polynomial bounded increase at infinity $\mathcal{O}_M$. The index i takes a finite number of values and the expressions $W_i(X)$, $W_{l;i}(X)$, $W_{l;i}^\mu(X)$ are Wick polynomials.

Using the linear independence one obtains from (3.2.5) a set of identities among Wick polynomials of the type

$$d_Q W_i = \cdots \tag{3.2.9}$$

where the left-hand side can be computed as follows. First one makes the derivation operations in the right-hand side of (3.2.5). It is quite possible that relations of the type

$$\frac{\partial}{\partial x_l^\mu} d_{l;i}^\mu(X) = \sum_j c_j d_j(X) \tag{3.2.10}$$

are valid for some numbers c_j. Then one has to rearrange the expression in the right-hand side of (3.2.5) and the right-hand side of (3.2.9) emerges as the coefficient of $d_i(X)$.

Identities of the type (3.2.10) are called *Ward–Takahashi* (or *Slavnov–Taylor identities*). In [12] these relations are called the *C-g identities*. They have been extensively studied in [9]. In lower orders of perturbation theory one can check them by explicit computation.

One now can interpret the renormalization theory as a distribution-splitting preservation of the Ward identities. Suppose that one can find a causal splitting $d_i = d_i^{\rm adv} - d_i^{\rm ret}$ of the set of causal distributions $d_i(X)$ such that we preserve Poincaré covariance, the order of singularity and the identities (3.2.10); i.e., we also have

$$\frac{\partial}{\partial x_l^\mu}(d_{l;i}^\mu)^{\rm adv(ret)}(X) = \sum_j c_j d_j^{\rm adv(ret)}(X). \tag{3.2.11}$$

Then we define the expressions $A(X)$ and $A_l^\mu(X)$ by making into the formulæ (3.2.8) the substitutions $d \to d^{\rm adv}$. If we use now the relations (3.2.9) we easily obtain

$$d_Q A(X) = {\rm i}\sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} A_l^\mu(X) \qquad |X| = n. \tag{3.2.12}$$

The similar property for the chronological products of order n easily follows. So, the obstructions to the gauge invariance in order n can appear in the process of causally splitting the relations (3.2.10) i.e. we might have instead of (3.2.11)

$$\frac{\partial}{\partial x_l^\mu}(d_{l;i}^\mu)^{\rm adv(ret)}(X) - \sum_j c_j d_j^{\rm adv(ret)}(X) = p(X) \tag{3.2.13}$$

where the expression in the right-hand side $p(X)$—called the *anomaly*—must have the form

$$p(X) = p(\partial)\delta(X) \tag{3.2.14}$$

where $p(\partial)$ is a Lorentz covariant polynomial in the partial derivative operators and

$$\delta(X) \equiv \delta(x_1 - x_n)\cdots\delta(x_{n-1} - x_n). \tag{3.2.15}$$

Also, if the distribution appearing in (3.2.10) has some global symmetry property (symmetry with respect to some global group of symmetries, (anti-) symmetry with respect to some indices etc) one can usually perform the distribution splitting such that these properties are also preserved. Moreover, we have a limitation on the degree of the polynomial $p(\partial)$:

$$\deg(p) \leqslant \omega \tag{3.2.16}$$

where ω is the order of singularity of the left-hand side of (3.2.13). There easily follows a case where there are no anomalies, namely when $\omega(d_i^\mu) \leqslant -2$, $\forall\mu$. Let us note in closing this section that the form of a anomaly can be simplified by redefinitions of the distributions a_i and $a_{l;i}^\mu$; we have the freedom of adding expressions of the type $p(\partial)\delta(X)$.

4. Second-order perturbation theory

4.1. Yang–Mills coupled to matter

We follow [17], where the pure Yang–Mills case was studied, emphasizing the possible appearance of anomalies in a more explicit way.

Theorem 4.1. *Suppose that the distribution $T(x, y)$ verifies (3.1.10). Then it verifies the formal adiabatic limit condition if and only if the following identities are verified:*

$$f_{abc} f_{dec} + f_{bdc} f_{aec} + f_{dac} f_{bec} = 0 \qquad a, b, d, e = 1, \ldots, r \tag{4.1.1}$$

$$f'_{dca} f'_{ceb} - f'_{dcb} f'_{cea} = -f_{abc} f'_{dec} \qquad a, b, d, e = 1, \ldots, r \tag{4.1.2}$$

$$f'_{cab} f''_{cde} + f'_{cdb} f''_{cae} + f'_{ceb} f''_{cda} = 0 \qquad \textit{iff} \quad m_b = 0 \tag{4.1.3}$$

$$\mathcal{S}_{bcdef} f'_{cba} g_{cdef} = 0 \qquad a, b, d, e, f = 1, \ldots, r \tag{4.1.4}$$

$$[t^\epsilon_a, t^\epsilon_b] = \mathrm{i} f_{abc} t^\epsilon_c \qquad \epsilon = \pm \quad a, b = 1, \ldots, r \tag{4.1.5}$$

$$t^-_a s^+_b - s^+_b t^+_a = \mathrm{i} f'_{bca} s^+_c \qquad a, b = 1, \ldots, r. \tag{4.1.6}$$

Here $\mathcal{S}_{\ldots}$ is the symmetrization operator in the indices which are explicitly exhibited.

Proof. (i) According to the ideas from section 3.2, we compute the commutator

$$D(x_1, x_2) \equiv [T(x_1), T(x_2)] \tag{4.1.7}$$

using the Wick theorem and identify a set of linearly independent distributions d_i as in (3.2.8); these are distributions in one variable $\xi \equiv x_1 - x_2$ due to translation invariance. Direct inspection of the expressions (2.1.14) and (2.2.1) produces a list of such distributions Δ with causal support. Using Feynman graph terminology we have distributions associated with tree and one-, two- and three-loop graphs. All these distributions can be written as sum of the positive (negative) frequency parts:

$$\Delta = \Delta^{(+)} + \Delta^{(-)}. \tag{4.1.8}$$

(a) From tree graphs:

$$\begin{aligned} & D_m \qquad \partial_\rho D_m \qquad \partial_\rho \partial_\sigma D_m \\ & S_M(x) \equiv (\mathrm{i}\gamma \cdot \partial + M) D_M(x) \end{aligned} \tag{4.1.9}$$

where D_m is the Pauli–Villars commutator distribution of causal support corresponding to mass m (see [17] for the definition) and S_M is the similar distribution for a Dirac field of mass M.

(b) From one-loop graphs we obtain new distributions with causal support:

$$\begin{aligned} & D^{(\pm)}_{m_1,m_2} \equiv \pm D^{(\pm)}_{m_1}(x) D^{(\pm)}_{m_2}(x) \\ & D^{(\pm)}_{m_1,m_2;\rho} \equiv \pm D^{(\pm)}_{m_1} \partial_\rho D^{(\pm)}_{m_2} - (1 \leftrightarrow 2) \\ & \partial_\rho D_{m_1,m_2} \\ & D^{(\pm)}_{m_1,m_2;\rho\sigma} \equiv \pm\left[\partial_\rho D^{(\pm)}_{m_1} \partial_\sigma D^{(\pm)}_{m_2} - D^{(\pm)}_{m_1} \partial_\rho \partial_\sigma D^{(\pm)}_{m_2}\right] + (1 \leftrightarrow 2) \\ & P^{(\pm)}_{M_1,M_2}(x) \equiv \pm \operatorname{Tr}\left[S^{(-)}_{M_1}(\mp x) S^{(+)}_{M_2}(\pm x)\right] \\ & P^{(\pm)}_{M_1,M_2;\rho}(x) \equiv \pm \operatorname{Tr}\left[S^{(-)}_{M_1}(\mp x) \gamma_\rho S^{(+)}_{M_2}(\pm x)\right] \\ & P^{(\pm)}_{M_1,M_2,\rho\sigma}(x) \equiv \pm \operatorname{Tr}\left[\gamma_\rho S^{(-)}_{M_1}(\mp x) \gamma_\sigma S^{(+)}_{M_2}(\pm x)\right] \\ & \Sigma^{(\pm)}_{m,M} \equiv \pm D^{(\pm)}_m S^{(\pm)}_M. \end{aligned} \tag{4.1.10}$$

We note that in the definition of $D^{(\pm)}_{m_1,m_2;\rho}$ we have taken the antisymmetric part in the masses because the symmetric part has been considered separately: it is the third distribution from the list.

(c) From two-loop graphs:

$$\begin{aligned} & D^{(\pm)}_{m_1,m_2,m_3} \equiv D^{(\pm)}_{m_1} D^{(\pm)}_{m_2} D^{(\pm)}_{m_3} \\ & \partial^2 D_{m_1,m_2,m_3} \\ & D^{(\pm)}_{m_1,m_2;m_3} \equiv \partial_\mu D^{(\pm)}_{m_1} \partial^\mu D^{(\pm)}_{m_2} D^{(\pm)}_{m_3} \\ & P^{(\pm)}_{m;M_1,M_2} \equiv \pm D^{(\pm)}_m P^{(\pm)}_{M_1,M_2} \\ & P^{(\pm)}_{m;M_1,M_2;\rho\sigma} \equiv \pm D^{(\pm)}_m P^{(\pm)}_{M_1,M_2;\rho\sigma}. \end{aligned} \tag{4.1.11}$$

(d) From three-loop graphs:

$$D^{(\pm)}_{m_1,m_2,m_3,m_4}(x) \equiv \pm D^{(\pm)}_{m_1} D^{(\pm)}_{m_2} D^{(\pm)}_{m_3} D^{(\pm)}_{m_4}. \tag{4.1.12}$$

The distributions $P^{\cdots}_{\cdots}$ are obtained from contractions of two vectorial currents. Let us note that one also obtains distributions of the type $Q^{\cdots}_{\cdots}$ from contractions of two axial currents. These distributions can be obtained directly from the corresponding distributions $P^{\cdots}_{\cdots}$ by conveniently inserting two γ_5 factors. However, the distributions of the type $Q^{\cdots}_{\cdots}$ can be expressed in terms of $P^{\cdots}_{\cdots}$ if one uses the identity

$$\gamma_5 S^{(\pm)}_M \gamma_5 = -S^{(\pm)}_{-M}. \tag{4.1.13}$$

The distributions following from contractions of an axial and a vectorial current are null because the traces so obtained are null. Next, we note that in the other commutators

$$D^\mu_1(x_1,x_2) \equiv [T^\mu(x_1), T(x_2)] \qquad D^\mu_2(x_1,x_2) \equiv [T(x_1), T^\mu(x_2)] = -D^\mu_1(x_2,x_1) \tag{4.1.14}$$

the distributions $g_{\mu\lambda} d^\lambda_{l;i}$ from (3.2.8) can be of the following type

$$\partial_\mu D_m \qquad \gamma_\mu S_M \qquad D_{m_1,m_2;\mu} \qquad D_{m_1,m_2;\mu\nu} \qquad P_{M_1,M_2;\mu} \qquad P_{M_1,M_2;\mu\nu} \tag{4.1.15}$$

and the distributions of the type d_i can be of the type

$$D_m \qquad \partial_\rho D_m \qquad D_{m_1,m_2} \qquad D_{m_1 m_2;\rho}. \tag{4.1.16}$$

Here the various parameters $m, M, \ldots$ are the masses appearing in the theory. If we consider distinct combinations of masses and indices we obtain a linear independent set of distributions.

Let us also give for further use the orders of singularity of the distributions listed above. We have

$$\begin{aligned}
&\omega(D_m) = -2 && \omega(D_{m_1,m_2}) = 0 && \omega(D_{m_1,m_2;\rho}) = -1\\
&\omega(D_{m_1,m_2;\rho\sigma}) = 2 && \omega(P_{M_1,M_2}) = 2 &&\\
&\omega(P_{M_1,M_2;\rho}) = 1 && \omega(P_{M_1,M_2;\rho\sigma}) = 2 &&\\
&\omega(P_{m;M_1,M_2}) = 4 && \omega(P_{m;M_1,M_2;\rho\sigma}) = 4 &&\\
&\omega(\Sigma_{m,M}) = 1 && \omega(D_{m_1,m_2,m_3}) = 2 &&\\
&\omega(D_{m_1,m_2;m_3}) = 4 && \omega(D_{m_1,m_2,m_3,m_4}) = 4. &&
\end{aligned} \tag{4.1.17}$$

Some of these orders of singularity are in fact lower than naive power counting suggests. All these distributions have causal support so we have causal decompositions

$$\Delta = \Delta^{\mathrm{adv}} - \Delta^{\mathrm{ret}}. \tag{4.1.18}$$

We have assumed that the causal splitting is preserving Lorentz covariance and the order of singularity. If the order of singularity is less 0 then this causal decomposition is unique (see the end of the preceding section). This is the case for the distributions D_m, S_M and $D_{m_1 m_2;\rho}$.

(ii) Now we consider the Ward identities (3.2.10). By direct inspection one finds out that they are

$$(\partial^2 + m^2) D_m = 0 \tag{4.1.19}$$

$$(\mathrm{i}\gamma\cdot\partial - M) S_M = S_M(\mathrm{i}\gamma\cdot \overleftarrow{\partial} - M) = 0 \tag{4.1.20}$$

$$\partial^\mu D_{m_1,m_2;\mu} = (m_2^2 - m_1^2) D_{m_1,m_2} \tag{4.1.21}$$

$$\partial^\mu D_{m_1,m_2;\mu\nu} = (m_2^2 - m_1^2) D_{m_1,m_2;\nu} \tag{4.1.22}$$

$$\partial^\mu P_{M_1,M_2;\mu} = \mathrm{i}(M_1 - M_2) P_{M_1,M_2} \tag{4.1.23}$$

$$\partial^\mu P_{M_1,M_2;\mu\nu} = \mathrm{i}(M_1 - M_2) P_{M_1,M_2;\nu}. \tag{4.1.24}$$

Now we analyse possible anomalies resulting after the causal splitting procedure. It is well known that the first two relations (4.1.19) and (4.1.20) indeed produce anomalies: for the (unique) causal splitting considered above one obtains

$$(\partial^2 + m^2) D_m^{\mathrm{adv(ret)}} = \delta \tag{4.1.25}$$

$$(\mathrm{i}\gamma \cdot \partial - M) S_M^{\mathrm{adv(ret)}} = S_M^{\mathrm{adv(ret)}} (\mathrm{i}\gamma \cdot \overleftarrow{\partial} - M) = -\delta. \tag{4.1.26}$$

One can prove more than that: even if we modify these splitting with arbitrary local polynomial terms the anomalies do not disappear.

Next we consider (4.1.22); inspecting the orders we can have the following generic form of the anomaly:

$$p_\nu(\partial) = c_1 \partial_\nu + c_3 \partial_\nu \partial^2. \tag{4.1.27}$$

We can eliminate this anomaly if we make the redefinition

$$D^{\mathrm{adv}}_{m_1,m_2;\mu\nu} \to D^{\mathrm{adv}}_{m_1,m_2;\mu\nu} + (c_1 g_{\mu\nu} + c_3 \partial_\mu \partial_\nu)\delta. \tag{4.1.28}$$

The case (4.1.24) can be treated in a similar way and the anomaly is also eliminated. The Ward identity (4.1.21) is non-trivial only for $m_1 \neq m_2$. We have already noticed that there exists a unique causal decomposition preserving Lorentz covariance and the order of singularity of $D_{m_1,m_2;\mu}$; then we can define

$$D^{\mathrm{adv}}_{m_1,m_2} = \frac{1}{m_2^2 - m_1^2} \partial^\mu D^{\mathrm{adv}}_{m_1,m_2;\mu} \tag{4.1.29}$$

and the relation (4.1.21) is preserved; moreover the order of singularity is preserved: $\omega(D^{\mathrm{adv}}_{m_1,m_2}) = \omega(D^{\mathrm{adv}}_{m_1,m_2;\rho}) + 1 = 0$.

The Ward identity (4.1.23) is non-trivial only for $M_1 \neq M_2$ and it has the generic form

$$p(\partial) = c_0 + c_2 \partial^2. \tag{4.1.30}$$

If we make the redefinitions

$$P^{\mathrm{adv}}_{M_1,M_2;\mu} \to P^{\mathrm{adv}}_{M_1,M_2;\mu} + c_2 \partial_\mu \delta \qquad P^{\mathrm{adv}}_{M_1,M_2} \to P^{\mathrm{adv}}_{M_1,M_2} + \mathrm{i}\frac{c_0}{M_1 - M_2}\delta \tag{4.1.31}$$

the anomaly is eliminated.

It is interesting to summarize the preceding argument by saying that the anomalies are produced only by the distributions associated with tree graphs.

(iii) It follows that we can describe the structure of the terms from $D_l^\mu(x_1, x_2)$ which can produce anomalies. It is sufficient to consider $l = 1$ and notice that the other part doubles the value of the anomaly (because of obvious symmetry properties). We have

$$\begin{aligned} D_1^\mu(x_1, x_2) = {} & \frac{\partial}{\partial x_{1\mu}} D_{m_c}(x_1 - x_2) T_c(x_1, x_2) + \frac{\partial^2}{\partial x_{1\mu} \partial x_1^\rho} D_{m_c}(x_1 - x_2) T_c^\rho(x_1, x_2) \\ & + \frac{\partial}{\partial x_{1\mu}} D_{m_c^*}(x_1 - x_2) T_c'(x_1, x_2) + \frac{\partial^2}{\partial x_{1\mu} \partial x_1^\rho} D_{m_c^*}(x_1 - x_2) T_c'^\rho(x_1, x_2) \\ & + \sum_{\alpha=1}^{8} : U_A^{(\alpha)}(x_1) \gamma^\mu S_A(x_1 - x_2) V_A^{(\alpha)}(x_2) \\ & + \sum_{\alpha=1}^{8} : T_A^{(\alpha)}(x_1) S_A(x_1 - x_2) \gamma^\mu W_A^{(\alpha)}(x_2) : + \cdots \end{aligned} \tag{4.1.32}$$

where by $\cdots$ we mean the contributions which do not produce anomalies because of the argument of (ii). We have the following explicit expressions:

$$\begin{aligned} &T_c(x_1,x_2) = T_c^{\mathrm{YM}}(x_1,x_2) + f_{abc} : u_a(x_1)A_b^\rho(x_1)j_{c\rho}(x_2) : \\ &T'_c(x_1,x_2) = T_c^{\prime YM}(x_1,x_2) - f'_{cab} : \Phi_a(x_1)u_b(x_1)j_c(x_2) : \\ &T_c^\rho(x_1,x_2) = T_c^{YM,\rho}(x_1,x_2) \qquad T_c^{\prime\rho}(x_1,x_2) = T_c^{\prime YM,\rho}(x_1,x_2) \end{aligned} \tag{4.1.33}$$

where the expressions $T_c^{\mathrm{YM}}(x_1,x_2)$, $T_c^{\prime YM}(x_1,x_2)$ and $T_c^{YM,\rho}(x_1,x_2)$, $T_c^{\prime YM,\rho}(x_1,x_2)$ correspond to the pure Yang–Mills case and can be found in [17]. Also

$$\begin{aligned} &U_A^{(1)}(x) = U_A^{(3)}(x) = U_A^{(5)}(x) = U_A^{(7)}(x) \equiv (t_a)_{BA}u_a(x)\overline{\psi}_B(x) \\ &U_A^{(2)}(x) = U_A^{(4)}(x) = U_A^{(6)}(x) = U_A^{(8)}(x) \equiv -(t'_a)_{BA}u_a(x)\overline{\psi}_B(x)\gamma_5 \\ &V_A^{(1)}(x) = V_A^{(4)}(x) \equiv (t_b)_{AD}\gamma_\rho\psi_D(x)A_b^\rho(x) \\ &V_A^{(2)}(x) = V_A^{(3)}(x) \equiv -(t'_b)_{AD}\gamma_\rho\gamma_5\psi_D(y)A_b^\rho(x) \\ &V_A^{(5)}(x) = V_A^{(8)}(x) \equiv (s_b)_{AD}\psi_D(x)\Phi_b(x) \\ &V_A^{(6)}(x) = V_A^{(7)}(x) \equiv (s'_b)_{AD}\gamma_5\psi_D(x)\Phi_b(x) \end{aligned} \tag{4.1.34}$$

and

$$\begin{aligned} &W_A^{(1)}(x) = W_A^{(3)}(x) = W_A^{(5)}(x) = W_A^{(7)}(x) \equiv (t_a)_{AB}u_a(x)\psi_B(x) \\ &W_A^{(2)}(x) = W_A^{(4)}(x) = W_A^{(6)}(x) = W_A^{(8)}(x) \equiv (t'_a)_{BA}u_a(x)\gamma_5\psi_B(x) \\ &T_A^{(1)}(x) = T_A^{(4)}(x) \equiv -(t_b)_{CA}\overline{\psi}_C(x)\gamma_\rho A_b^\rho(x) \\ &T_A^{(2)}(x) = T_A^{(3)}(x) \equiv -(t'_b)_{CA}\overline{\psi}_C(x)\gamma_\rho\gamma_5 A_b^\rho(x) \\ &T_A^{(5)}(x) = T_A^{(8)}(x) \equiv -(s_b)_{CA}\overline{\psi}_C(x)\Phi_b(x) \\ &T_A^{(6)}(x) = T_A^{(7)}(x) \equiv -(s'_b)_{CA}\overline{\psi}_C(x)\gamma_5\Phi_b(x). \end{aligned} \tag{4.1.35}$$

The expression of the anomaly can be obtained in the generic form

$$A(x_1,x_2) = \mathrm{i}\delta(x_1-x_2)A(x_1) \tag{4.1.36}$$

where

$$\begin{aligned} A(x_1) \equiv \sum_c &\left[T_c(x_1,x_1) + T'_c(x_1,x_1) - \left(\frac{\partial}{\partial x_1^\rho}T_c^\rho\right)(x_1,x_1) - \left(\frac{\partial}{\partial x_1^\rho}T_c^{\prime\rho}\right)(x_1,x_1)\right] \\ &+\mathrm{i}\sum_\alpha [: U_A^{(\alpha)}(x_1)V_A^{(\alpha)}(x_1) : + : T_A^{(\alpha)}(x_1)W_A^{(\alpha)}(x_1) :]. \end{aligned} \tag{4.1.37}$$

So, the expression of the anomaly $A(x)$ obtains an extra term because of the presence of the Dirac fermions:

$$\begin{aligned} A(x) = A^{\mathrm{YM}}(x) &+ \mathrm{i} : u_a(x)A_b^\rho(x)\overline{\psi}_A(x)\gamma_\rho([t_a,t_b] + [t'_a,t'_b] - \mathrm{i}f_{abc}t_c)_{AB}\psi_B(x) : \\ &+\mathrm{i} : u_a(x)A_b^\rho(x)\overline{\psi}_A(x)\gamma_\rho\gamma_5([t_a,t'_b] + [t'_a,t_b] - \mathrm{i}f_{abc}t'_c)_{AB}\psi_B(x) : \\ &+\mathrm{i} : u_a(x)\Phi_b(x)\overline{\psi}_A(x)([t_a,s_b] - \{t'_a,s'_b\} + \mathrm{i}f'_{cba}s_c)_{AB}\psi_B(x) : \\ &+\mathrm{i} : u_a(x)\Phi_b(x)\overline{\psi}_A(x)\gamma_5([t_a,s'_b] - \{t'_a,s_b\} + \mathrm{i}f'_{cba}s'_c)_{AB}\psi_B(x) : . \end{aligned} \tag{4.1.38}$$

(iv) We proceed now as in [17]. First we equate the expression $A(x)$ to a coboundary $d_Q L(x)$.

We obtain all the relations from [17] (and this explains the first four relations from the statement). Moreover we obtain for all $a, b = 1, \ldots, r$

$$\begin{aligned} &[t_a,t_b] + [t'_a,t'_b] = \mathrm{i}f_{abc}t_c \qquad &[t_a,t'_b] + [t'_a,t_b] = \mathrm{i}f_{abc}t'_c \\ &[t_a,s_b] - \{t'_a,s'_b\} = -\mathrm{i}f'_{cba}s_c \qquad &[t_a,s'_b] - \{t'_a,s_b\} = -\mathrm{i}f'_{cba}s'_c \end{aligned} \tag{4.1.39}$$

which are equivalent to the last two relations from the statement.

(v) From the preceding computations we can obtain the explicit expression for the coboundary $L(x)$: it coincides with the expression obtained for the pure Yang–Mills case:

$$\begin{aligned} L(x) = L^{\mathrm{YM}}(x) \equiv \tfrac{1}{4} f_{cab} f_{cde} : A_{a\nu}(x) A_{b\nu}(x) A_d^{\mu}(x) A_{\mathrm{e}}^{\nu}(x) : \\ - f'_{cda} f'_{ceb} : A_{a\nu}(x) A_b^{\nu}(x) \Phi_d(x) \Phi_{\mathrm{e}}(x) : \\ - \sum_{m_b \neq 0} g'_{abcd} : \Phi_a(x) \Phi_b(x) \Phi_d(x) \Phi_{\mathrm{e}}(x) : \end{aligned} \tag{4.1.40}$$

where

$$g'_{abcd} \equiv \frac{1}{2m_b} \mathcal{S}_{abde} f'_{cab} f''_{cde}. \tag{4.1.41}$$

Let us also define

$$L^{\mu}(x) \equiv \sum_c \left[T_c^{\mu}(x,x) + T_c'^{\mu}(x,x) \right]. \tag{4.1.42}$$

Again it coincides with the expression from the pure Yang–Mills case:

$$\begin{aligned} L^{\mu}(x) = L^{YM,\mu}(x) = - f_{cab} f_{cde} : u_a(x) A_{b\nu}(x) A_d^{\nu}(x) A_{\mathrm{e}}^{\mu}(x) : \\ - f'_{cab} f'_{cde} : \Phi_a(x) u_b(x) \Phi_d(x) A_{\mathrm{e}}^{\mu}(x) : . \end{aligned} \tag{4.1.43}$$

We consider now a *canonical* causal splitting $A^c(x_1, x_2)$ and $A_l^{c,\mu}(x_1, x_2)$ given by the expressions which are obtained from the corresponding commutators if we make the substitutions $\Delta \to \Delta^{\mathrm{adv}}$. This indeed gives a causal splitting of $D(x_1, x_2)$ and $D_l^{\mu}(x_1, x_2)$ respectively. However the identity (3.2.12) is not fulfilled. If we define now the new causal splitting

$$\begin{aligned} A(x_1, x_2) \equiv A^c(x_1, x_2) + \delta(x_1 - x_2) L(x_1) \\ A_l^{\mu}(x_1, x_2) \equiv A_l^{c,\mu}(x_1, x_2) + \delta(x_1 - x_2) L^{\mu}(x_1) \end{aligned} \tag{4.1.44}$$

then one can see that (3.2.12) becomes true. Moreover, in this way one can obtain in the usual way the expression of the chronological products $T(x_1, x_2)$ and $T_l^{\mu}(x_1, x_2)$ such that we have (3.2.4) and all other properties, in particular symmetry. □

Remark 4.2. If we do not require that (3.1.10) is fulfilled, the relations (4.1.4) and (4.1.6) acquire a weaker form.

The group-theoretical information contained in this theorem is:

(a) The expressions f_{abc} are the structure constants of a Lie algebra $\mathfrak{g}$.

(b) The structure constants f_{abc} corresponding to $m_a = m_b = m_c = 0$ generate a Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$.

(c) The $r \times r$ (antisymmetric) matrices T_a, $a = 1, \ldots, r$ defined according to

$$(T_a)_{bc} \equiv - f'_{bca} \qquad \forall a, b, c = 1, \ldots, r \tag{4.1.45}$$

are an r-dimensional representation of the Lie algebra $\mathfrak{g}$.

The representation T_a exhibited in the statement of the theorem is nothing else but the representation of the gauge algebra $\mathfrak{g}$ in which the Higgs fields live.

(d) The relation (4.1.5) tells us that the matrices t_a^{ϵ} are representations of the Lie algebra $\mathfrak{g}$ and relation (4.1.6) shows that the matrices s_a^{ϵ} are some tensor operators with respect to the couple of representations t_b^{ϵ} of the Lie algebra $\mathfrak{g}$.

So, we propose the following strategy of analysing the generalization of the SM described in this paper: first one should find restrictions on the Lie algebra $\mathfrak{g}$ from the relation (4.1.2), then one takes a couple of representations t_a^ϵ of this Lie algebra and afterwards one determines the matrices s_a^+ from the relation (4.1.6) using ideas from the proof of the Wigner–Eckart theorem. We mention that if one tries to substitute the formula (2.2.18) into the formula (4.1.6), as done in [3], then we end up with some very complicated trilinear relations, which are extremely difficult to analyse in the general case.

Next, we have a generalization of proposition 3.9 from [17]. By definition the *Feynman propagator* and the *Feynman antipropagator* are

$$\Delta^F \equiv \Delta^{\rm adv} - \Delta^{(-)} = \Delta^{\rm ret} + \Delta^{(+)} \qquad \Delta^{AF} \equiv \Delta^{(+)} - \Delta^{\rm adv} = -\Delta^{\rm ret} - \Delta^{(-)}. \tag{4.1.46}$$

Then we have:

Proposition 4.3. *Suppose that that there is no contribution $T_{1,\rm matter}$ in the first-order chronological product. Then, we have*

$$\begin{aligned}
T^c(x,y) = T^{YM,c}(x,y) \\
&-f_{abc} D^F_{m_c}(x-y)[:A_{a\nu}(x) F_b^{\nu\rho}(x) j_{c\rho}(y): \\
&-:u_a(x)\partial_\rho \tilde{u}_b(x) j_c^\rho(y): + (x \leftrightarrow y)] \\
&-f_{abc}\frac{\partial}{\partial x^\mu} D^F_{m_c}(x-y)[:A_a^\rho(x) A_b^\mu(x) j_{c\rho}(y): - (x \leftrightarrow y)] \\
&-f'_{abc} D^F_{m_c}(x-y)[:\Phi_a(x)\partial_\mu \Phi_b(x) j_c^\mu(y): - (x \leftrightarrow y)] \\
&-f'_{abc} D^F_{m_c^*}(x-y)[:\partial_\mu \Phi_a(x) A_b^\mu(x) j_c(y): + (x \leftrightarrow y)] \\
&-f'_{abc}\frac{\partial}{\partial x^\mu} D^F_{m_c^*}(x-y)[:\Phi_a(x) A_b^\mu(x) j_c(y): - (x \leftrightarrow y)] \\
&-2h^{(1)}_{abc} D^F_{m_c}(x-y)[:\Phi_a(x) A_b^\mu(x) j_{c\mu}(y): + (x \leftrightarrow y)] \\
&+h^{(1)}_{cab} D^F_{m_c^*}(x-y)[:A_{a\mu}(x) A_b^\mu(x) j_c(y): + (x \leftrightarrow y)] \\
&+h^{(2)}_{cab} D^F_{m_c^*}(x-y)[:\tilde{u}_a(x) u_b(x) j_c(y): + (x \leftrightarrow y)] \\
&+3h^{(3)}_{abc} D^F_{m_c^*}(x-y)[:\Phi_a(x)\Phi_b(x) j_c(y): + (x \leftrightarrow y)] \\
&+4g_{abcd} D^F_{m_c^*}(x-y)[:\Phi_a(x)\Phi_b(x)\Phi_c(x) j_c(y): + (x \leftrightarrow y)] \\
&+:A_a^\mu(x) A_b^\rho(y):\{[(t_a)_{AC}(t_b)_{CB}:\overline{\psi}_A(x)\gamma_\mu S^F_{M_C}(x-y)\gamma_\rho \psi_B(y): \\
&+(t'_a)_{AC}(t'_b)_{CB}:\overline{\psi}_A(x)\gamma_\mu\gamma_5 S^F_{M_C}(x-y)\gamma_\rho\gamma_5 \psi_B(y): \\
&+(t_a)_{AC}(t'_b)_{CB}:\overline{\psi}_A(x)\gamma_\mu S^F_{M_C}(x-y)\gamma_\rho\gamma_5 \psi_B(y): \\
&+(t'_a)_{AC}(t_b)_{CB}:\overline{\psi}_A(x)\gamma_\mu\gamma_5 S^F_{m_C}(x-y)\gamma_\rho \psi_B(y): - (a \leftrightarrow b, \mu \leftrightarrow \rho, x \leftrightarrow y)] \\
&+(t_a)_{AB}(t_b)_{BA} P^F_{M_A M_B;\mu\rho}(x-y) \\
&+(t'_a)_{AB}(t'_b)_{BA} Q^F_{M_A M_B;\mu\rho}(x-y)\} \\
&+:\Phi_a(x)\Phi_b(y):\{[(s_a)_{AC}(s_b)_{CB}:\overline{\psi}_A(x) S^F_{M_C}(x-y)\psi_B(y): \\
&+(s'_a)_{AC}(s'_b)_{CB}:\overline{\psi}_A(x)\gamma_5 S^F_{M_C}(x-y)\gamma_5 \psi_B(y): \\
&+(s_a)_{AC}(s'_b)_{CB}:\overline{\psi}_A(x) S^F_{M_C}(x-y)\gamma_5 \psi_B(y): \\
&+(s'_a)_{AC}(s_b)_{CB}:\overline{\psi}_A(x)\gamma_5 S^F_{M_C}(x-y)\psi_B(y): - (a \leftrightarrow b, x \leftrightarrow y)] \\
&+(s_a)_{AB}(s_b)_{BA} P^F_{M_A,M_B}(x-y) + (s'_a)_{AB}(s'_b)_{BA} Q^F_{M_A,M_B}(x-y)\} \\
&+:A_a^\mu(x)\Phi_b^\rho(y):\{[(t_a)_{AC}(s_b)_{CB}:\overline{\psi}_A(x)\gamma_\mu S^F_{M_C}(x-y)\psi_B(y):
\end{aligned}$$

$$
\begin{aligned}
&-(s_b)_{AC}(t_a)_{CB} : \overline{\psi}_A(y) S^F_{M_C}(y-x)\gamma_\mu \psi_B(x) : \\
&+(t'_a)_{AC}(s'_b)_{CB} : \overline{\psi}_A(x)\gamma_\mu\gamma_5 S^F_{M_C}(x-y)\gamma_5\psi_B(y) : \\
&-(s'_b)_{AC}(t'_a)_{CB} : \overline{\psi}_A(y)\gamma_5 S^F_{M_C}(y-x)\gamma_\mu\gamma_5\psi_B(x) : \\
&+(t_a)_{AC}(s'_b)_{CB} : \overline{\psi}_A(x)\gamma_\mu S^F_{M_C}(x-y)\gamma_5\psi_B(y) : \\
&-(s'_b)_{AC}(t_a)_{CB} : \overline{\psi}_A(y)\gamma_5 S^F_{M_C}(y-x)\gamma_\mu\psi_B(x) : \\
&+(t'_a)_{AC}(s_b)_{CB} : \overline{\psi}_A(x)\gamma_\mu S^F_{M_C}(x-y)\gamma_5\psi_B(y) : \\
&-(s_b)_{AC}(t'_a)_{CB} : \overline{\psi}_A(y)\gamma_5 S^F_{M_C}(y-x)\gamma_\mu\psi_B(x) : \\
&+(t_a)_{AB}(s_b)_{BA} P^F_{M_A,M_B;\mu}(x-y) \\
&+(t'_a)_{AB}(s'_b)_{BA} Q^F_{M_A,M_B;\mu}(x-y)] - [x \leftrightarrow y]\} \\
&-D^F_{m_a}(x-y) : j_{a\mu}(x) j^\mu_a(y) : \\
&-(t_a)_{AC}(t_a)_{CB}[: \overline{\psi}_A(x)\gamma_\mu \Sigma^F_{m_a,M_C}(x-y)\gamma^\mu\psi_B(y) : +(x\leftrightarrow y)] \\
&-(t'_a)_{AC}(t'_a)_{CB}[: \overline{\psi}_A(x)\gamma_\mu\gamma_5 \Sigma^F_{m_a,M_C}(x-y)\gamma^\mu\gamma_5\psi_B(y) : +(x\leftrightarrow y)] \\
&-(t_a)_{AC}(t'_a)_{CB}[: \overline{\psi}_A(x)\gamma_\mu \Sigma^F_{m_a,M_C}(x-y)\gamma^\mu\gamma_5\psi_B(y) : +(x\leftrightarrow y)] \\
&-(t'_a)_{AC}(t_a)_{CB}[: \overline{\psi}_A(x)\gamma_\mu\gamma_5 \Sigma^F_{m_a,M_C}(x-y)\gamma^\mu\psi_B(y) : +(x\leftrightarrow y)] \\
&-g^{\mu\nu}[(t_a)_{AB}(t_a)_{BA} P^F_{m_a;M_A,M_B;\mu\nu}(x-y) \\
&+(t'_a)_{AB}(t'_a)_{BA} Q^F_{m_a;M_A,M_B;\mu\nu}(x-y)] \\
&+D^F_{m^*_a}(x-y) : j_a(x) j_a(y) : \\
&+(s_a)_{AC}(s_a)_{CB}[: \overline{\psi}_A(x)\Sigma^F_{m_a,M_C}(x-y)\psi_B(y) : +(x\leftrightarrow y)] \\
&+(s'_a)_{AC}(s'_a)_{CB}[: \overline{\psi}_A(x)\gamma_5\Sigma^F_{m_a,M_C}(x-y)\gamma_5\psi_B(y) : +(x\leftrightarrow y)] \\
&+(s_a)_{AC}(s'_a)_{CB}[: \overline{\psi}_A(x)\Sigma^F_{m_a,M_C}(x-y)\gamma_5\psi_B(y) : +(x\leftrightarrow y)] \\
&+(s'_a)_{AC}(s_a)_{CB}[: \overline{\psi}_A(x)\gamma_5\Sigma^F_{m_a,M_C}(x-y)\psi_B(y) : +(x\leftrightarrow y)].
\end{aligned} \tag{4.1.47}
$$

Here $h^{(1)}_{abc} \equiv \frac{1}{2}(f'_{bca}m_b + f'_{acb}m_a)$ *and* $h^{(2)}_{abc} \equiv f'_{abc}m_b$.

Let us note that the expressions (2.2.12) and (2.2.13) for the currents can be also written as follows:

$$j^\mu_a(x) =: \overline{\psi^+_A}(x)(t^+_a)_{AB}\gamma^\mu\psi^+_B(x) : + : \overline{\psi^-_A}(x)(t^-_a)_{AB}\gamma^\mu\psi^-_B(x) : \tag{4.1.48}$$

and

$$j_a(x) =: \overline{\psi^-_A}(x)(s^+_a)_{AB}\psi^+_B(x) : + : \overline{\psi^+_A}(x)(s^-_a)_{AB}\psi^-_B(x) : \tag{4.1.49}$$

where we have defined

$$\psi^\epsilon_A(x) \equiv \frac{1+\epsilon\gamma_5}{2}\psi_A(x) \qquad \epsilon = \pm \tag{4.1.50}$$

and the components corresponding to the signs + (−) are called *chiral* components of the currents.

4.2. The conservation of the BRST current

The expression

$$j^\mu_{\rm BRST}(x) \equiv (\partial\cdot A_a + m_a\Phi_a) \overset{\leftrightarrow\mu}{\partial} u_a \tag{4.2.1}$$

is called the *BRST current*. One can verify easily the *conservation of the BRST current*:

$$\partial_\mu j^\mu_{\mathrm{BRST}} = 0. \tag{4.2.2}$$

Formally, the BRST charge is given by

$$Q = \int_{\mathbb{R}^3} \mathrm{d}^3x \; j^0_{\mathrm{BRST}}(x). \tag{4.2.3}$$

We want to investigate the conservation of this current in higher orders of perturbation theory. We present here the analysis in the second order. First we have:

Proposition 4.4. *The following relation is verified:*

$$\begin{aligned}[]
[j^\mu_{\mathrm{BRST}}(x_1), T(x_2)] = {} & D_{m_a}(x_1 - x_2) A^\mu_a(x_1, x_2) + \partial^\mu D_{m_a}(x_1 - x_2) B_a(x_1, x_2) \\
& + \partial^\rho D_{m_a}(x_1 - x_2) A^{\mu\rho}_a(x_1, x_2) + \partial^\mu \partial_\rho D_{m_a}(x_1 - x_2) B^\rho_a(x_1, x_2)
\end{aligned} \tag{4.2.4}$$

where

$$\begin{aligned}
B_a(x_1, x_2) = {} & h^{(2)}_{bac} : \partial_\nu A^\nu_a(x_1) \Phi_b(y) u_c(x_2) : + m_a h^{(2)}_{bac} : \Phi_a(x_1) \Phi_b(y) u_c(x_2) : \\
& - m_a f'_{abc} : u_a(x_1) \partial_\rho \Phi_b(x_2) A^\rho_c(x_2) : - m_a h^{(1)}_{abc} : u_a(x_1) A_{b\rho}(x_2) A^\rho_c(x_2) : \\
& - m_a h^{(2)}_{abc} : u_a(x_1) \tilde{u}_b(x_2) u_c(x_2) : \\
& - 3 m_a f''_{bca} : u_a(x) \Phi_b(y) \Phi_c(y) : - m_a : u_a(x_1) j_b(x_2) :
\end{aligned} \tag{4.2.5}$$

and

$$\begin{aligned}
B^\rho_a(x_1, x_2) = {} & f_{bca} : \partial_\nu A^\nu_a(x_1) A^\rho_b(x_2) u_c(x_2) : - m_a f_{bca} : \Phi_a(x_1) A^\rho_b(x_2) u_c(x_2) : \\
& + f_{abc} : u_a(x_1) A_{b\nu}(x_2) F^{\nu\rho}_c(x_2) : - f_{abc} : u_a(x_1) u_b(x_2) \partial^\rho \tilde{u}_c(x_2) : \\
& + f'_{bca} : u_a(x_1) \Phi_b(x_2) \partial^\rho \Phi_c(x_2) : \\
& + m_c f'_{cba} : u_a(x_1) \Phi_b(x_2) A^\rho_c(x_2) : + : u_a(x_1) j^\rho_a(x_2) : .
\end{aligned} \tag{4.2.6}$$

The expressions for $h^{(1)}_{cab}$ and $h^{(2)}_{abc}$ have been given in the preceding proposition.

The computations are long but straightforward. Applying the procedures of the preceding subsection we obtain from here:

Proposition 4.5. *The expression*

$$\begin{aligned}
T^c(j^\mu_{\mathrm{BRST}}(x_1), T(x_2)) = {} & D^F_{m_a}(x_1 - x_2) A^\mu_a(x_1, x_2) + \partial^\mu D^F_{m_a}(x_1 - x_2) B_a(x_1, x_2) \\
& + \partial^\rho D^F_{m_a}(x_1 - x_2) A^{\mu\rho}_a(x_1, x_2) + \partial^\mu \partial_\rho D^F_{m_a}(x_1 - x_2) B^\rho_a(x_1, x_2)
\end{aligned} \tag{4.2.7}$$

is valid for the canonical chronological product.

We have the following result which can be interpreted as a conservation of the BRST current in the second order of perturbation theory.

Theorem 4.6. *There exists a finite renormalization such that one has the following conservation law:*

$$\frac{\partial}{\partial x^\mu_1} T(j^\mu_{\mathrm{BRST}}(x_1), T(x_2)) = \mathrm{i} \left(\frac{\partial}{\partial x^\mu_1} \delta(x_1 - x_2) \right) T^\mu(x_1). \tag{4.2.8}$$

Proof. We start from the obvious relation

$$\frac{\partial}{\partial x^\mu_1} [(j^\mu_{\mathrm{BRST}}(x_1), T(x_2)] = 0 \tag{4.2.9}$$

and perform the canonical causal splitting using the expression of the commutator derived above. If we proceed in analogy to the derivation of the consistency conditions for the second-order chronological products we obtain

$$\frac{\partial}{\partial x_1^\mu} T^c(j^\mu_{\mathrm{BRST}}(x_1), T(x_2)) = -\mathrm{i}\frac{\partial}{\partial x_1^\mu}\left[\delta(x_1 - x_2)N^\mu(x_1)\right] - \mathrm{i}\delta(x_1 - x_2)A(x_1) \tag{4.2.10}$$

where

$$A(x_1) \equiv \sum_a \left[B_a(x_1, x_1) - \left(\frac{\partial B_a^\mu}{\partial x_1^\mu}\right)(x_1, x_1)\right] \tag{4.2.11}$$

and

$$N^\mu(x_1) \equiv \sum_a B_a^\mu(x_1, x_1). \tag{4.2.12}$$

Now it is a matter of computation to prove that we have $A = -\partial_\mu T^\mu$. If we perform the finite renormalization $T(j^\mu_{\mathrm{BRST}}(x_1), T(x_2)) = T^c(j^\mu_{\mathrm{BRST}}(x_1), T(x_2)) - \mathrm{i}\delta(x_1 - x_2)\,[N^\mu(x_1) + T^\mu(x)]$ then we obtain the relation from the statement. □

Remark 4.7. If we perform the finite renormalization

$$T'(j^\mu_{\mathrm{BRST}}(x_1), T(x_2)) = T^c(j^\mu_{\mathrm{BRST}}(x_1), T(x_2)) - \mathrm{i}\delta(x_1 - x_2)N^\mu(x_1)$$

then we obtain the relation

$$\frac{\partial}{\partial x_1^\mu} T'(j^\mu_{\mathrm{BRST}}(x_1), T(x_2)) = \mathrm{i}\delta(x_1 - x_2)\frac{\partial}{\partial x_1^\mu} T^\mu(x_1). \tag{4.2.13}$$

Using the method of appendix B of [10] (where the case of QED is investigated) we can obtain from (4.2.8) that the BRST current is conserved if the coupling constant (a test function) is constant in a neighbourhood of the point x.

5. Third-order gauge invariance

5.1. The derivation of the anomaly

In this section we will analyse the possible obstructions to factorization of the S-matrix in the third order of the perturbation theory. In principle, there is no difference with respect to the preceding section. Nevertheless, the details of distribution splitting are considerably more complicated and the same is true for the whole combinatorial argument.

First we give a standard regularization procedure of the distributions appearing in the lists (4.1.9)–(4.1.12). We choose $m_0 > 0$ different from all masses of the model and write the Pauli–Villars distribution for any mass as follows:

$$D_m = D_{m_0} + D^{\mathrm{reg}}; \tag{5.1.1}$$

one can check that the order of singularity of D^{reg} is

$$\omega(D^{\mathrm{reg}}) = -4. \tag{5.1.2}$$

The decomposition (5.1.1) induces a similar decomposition for all distributions in the lists (4.1.9)–(4.1.12): we have a sum of two pieces

$$\Delta = \Delta^0 + \Delta^{\mathrm{reg}} \tag{5.1.3}$$

where

$$\omega(\Delta^0) = \omega(\Delta) \qquad \omega(\Delta^{\mathrm{reg}}) = \omega(\Delta) - 2 \tag{5.1.4}$$

and the support properties of Δ^0 in the momentum space are more convenient. We have

$$\tilde{\Delta}^{0,(\pm)}(p) \sim \theta(\pm p_0) f(p^2) \tag{5.1.5}$$

with $\mathrm{supp}(f) \subset \{p^2 \geqslant \lambda^2\}$ for some parameter with mass significance $\lambda > 0$ (for the distributions $\tilde{\Delta}^{(\pm)}(p)$ we can have in principle $\lambda = 0$).

The main result is contained in the following theorem:

Theorem 5.1. *Suppose that the distribution $T(x_1, x_2, x_3)$ verifies the condition (3.1.10). Then it verifies the formal adiabatic limit condition if and only if, beside the conditions from the statement of theorem 2.1, we also have the following set of supplementary conditions:*

$$\mathrm{Tr}\left(t_a^+\{t_b^+, t_c^+\}\right) = \mathrm{Tr}\left(t_a^-\{t_b^-, t_c^-\}\right) \tag{5.1.6}$$

$$f'_{abc}g'_{bfgh} + f'_{fbc}g'_{bagh} + f'_{gbc}g'_{bafh} + f'_{hbc}g'_{bafg} = 0. \tag{5.1.7}$$

Proof. (i) As before, we will investigate the third-order commutators

$$D(x_1, x_2; x_3) = [T(x_3), \overline{T}(x_1, x_2)] - [T(x_1, x_3), \overline{T}(x_2)] - [T(x_2, x_3), \overline{T}(x_1)] \tag{5.1.8}$$

and

$$\begin{aligned} D_1^\mu(x_1, x_2; x_3) &= [T(x_3), \overline{T}_1^\mu(x_1, x_2)] - [T_1^\mu(x_1, x_3), \overline{T}(x_2)] - [T(x_2, x_3), \overline{T}_1^\mu(x_1)] \\ D_2^\mu(x_1, x_2; x_3) &= [T(x_3), \overline{T}_2^\mu(x_1, x_2)] - [T(x_1, x_3), \overline{T}_1^\mu(x_2)] - [T_1^\mu(x_2, x_3), \overline{T}(x_1)] \\ D_3^\mu(x_1, x_2; x_3) &= [T_1^\mu(x_3), \overline{T}(x_1, x_2)] - [T_2^\mu(x_1, x_3), \overline{T}(x_2)] - [T_2^\mu(x_2, x_3), \overline{T}(x_1)]. \end{aligned} \tag{5.1.9}$$

All these operator-valued distributions have the causal support property.

(ii) We convene to denote generically by

$$\begin{aligned} \Delta_3^{(+)}(x_1 - x_2) &= \prod_i \langle \Omega, \phi_i(x_1)\psi_i(x_2)\Omega \rangle \\ \Delta_1^{(+)}(x_2 - x_3) &= \prod_j \langle \Omega, \phi_j(x_2)\chi_j(x_3)\Omega \rangle \\ \Delta_2^{(+)}(x_3 - x_1) &= \prod_k \langle \Omega, \psi_k(x_3)\chi_k(x_1)\Omega \rangle \end{aligned} \tag{5.1.10}$$

the distributions appearing in the analysis of the second-order perturbation theory i.e. the lists (4.1.9)–(4.1.12). They appear with these three combinations of arguments from various Wick contractions in the preceding formulæ for the commutators. Here the fields $\phi(x_1)$, $\psi(x_2)$, $\chi(x_3)$) are factors in the Wick monomials of $T(x_1)$, $T(x_2)$, $T(x_3)$ respectively. If Fermi fields are present one has to take into account the signs induced by the permutation of the non-commuting factors in defining the associated distributions $\Delta^{(-)}$.

We have to investigate the types of numerical distribution with causal support which can appear from the computation of the four commutators. These distributions will depend only on two variables $\xi_1 \equiv x_1 - x_3$, $\xi_2 \equiv x_2 - x_3$ due to translation invariance. It convenient to use again a graph theory terminology. We define a *super-line* to be the assemble of lines of a Feynman graph connecting two vertices. Then the notions of *super-tree* and *super-loop* are obvious and we have only such types of graph. We give the generic form of the distributions associated with them.

(a) First we obtain some distributions containing a factor δ from commutators containing a factor $\delta(x-y)L(x)$ or $\delta(x-y)L^\mu(x)$. In this case we obtain distributions of the type

$$d(\Delta)(x_1, x_2; x_3) = \delta(x_1 - x_2)\Delta(x_2 - x_3) \tag{5.1.11}$$

and other permutations of the variables. Here the distribution Δ is one from the lists (4.1.9)–(4.1.12).

(b) Next, from super-tree graphs we obtain three types of distribution.

(b1) There exists a super-line between x_1 and x_3 and a super-line between x_2 and x_3. In this case one obtains distributions of the form

$$d_{(3)}(x_1, x_2; x_3) = \Delta_1^{(+)}(x_2 - x_3)\Delta_2^{(-)}(x_3 - x_1) - \Delta_1^{(-)}(x_2 - x_3)\Delta_2^{(+)}(x_3 - x_1)$$
$$+\Delta_2^F(x_3 - x_1)\Delta_1(x_2 - x_3) - \Delta_1^F(x_2 - x_3)\Delta_2(x_3 - x_1). \quad (5.1.12)$$

The causal support of this type of distribution can be checked if one derives alternative formulæ. If

$$d_{(3)}^{\mathrm{adv(ret)}}(x_1, x_2; x_3) = \Delta_2^{\mathrm{ret(adv)}}(x_3 - x_1)\Delta_1^{\mathrm{adv(ret)}}(x_2 - x_3) \quad (5.1.13)$$

then we have from (4.1.46)

$$d_{(3)}(x_1, x_2; x_3) = d_{(3)}^{\mathrm{adv}}(x_1, x_2; x_3) - d_{(3)}^{\mathrm{ret}}(x_1, x_2; x_3). \quad (5.1.14)$$

Moreover, if one uses the expression of the third-order chronological product (3.1.22) one can prove that the distribution of this type produces the Feynman propagator

$$d_{(3)}^F(x_1, x_2; x_3) = \Delta_2^F(x_3 - x_1)\Delta_1^F(x_2 - x_3). \quad (5.1.15)$$

(b2) There exists a super-line between x_1 and x_2 and a super-line between x_1 and x_3. In this case one obtains distributions of the form

$$d_{(1)}(x_1, x_2; x_3) = \Delta_2^{(+)}(x_3 - x_1)\Delta_3^{(-)}(x_1 - x_2) - \Delta_2^{(-)}(x_3 - x_1)\Delta_3^{(+)}(x_1 - x_2)$$
$$+\Delta_3^{AF}(x_1 - x_2)\Delta_2(x_3 - x_1) - \Delta_2^F(x_3 - x_1)\Delta_3(x_1 - x_2). \quad (5.1.16)$$

The causal support of this type of distribution can be also checked if one derives the alternative formulæ. We define

$$d_{(1)}^{\mathrm{adv(ret)}}(x_1, x_2; x_3) = \Delta_3^{\mathrm{ret(adv)}}(x_1 - x_2)\Delta_1^{\mathrm{ret(adv)}}(x_3 - x_1) \quad (5.1.17)$$

and we have as before

$$d_{(1)}(x_1, x_2; x_3) = d_{(1)}^{\mathrm{adv}}(x_1, x_2; x_3) - d_{(1)}^{\mathrm{ret}}(x_1, x_2; x_3). \quad (5.1.18)$$

If one uses the expression of the third-order chronological product (3.1.22) one can prove that the distribution of this type produces the Feynman propagator

$$d_{(1)}^F(x_1, x_2; x_3) = \Delta_3^F(x_1 - x_2)\Delta_2^F(x_3 - x_1). \quad (5.1.19)$$

(b3) There exists a super-line between x_1 and x_2 and a super-line between x_2 and x_3. In this case one obtains distributions $d_{(2)}(x_1, x_2; x_3)$ of the same form as in case (b2) if one makes $x_1 \leftrightarrow x_2$.

We will denote the distributions associated with super-tree graphs by $d_{(i)}(\Delta, \Delta')(x_1, x_2; x_3)$, indicating explicitly the distributions in one variable Δ, Δ' from the lists (4.1.9)–(4.1.12) involved in the construction. One can verify that if the orders of singularity of these distributions are ω and ω' respectively, then

$$\omega(d_{(i)}(\Delta, \Delta')) = 4 + \omega + \omega'. \quad (5.1.20)$$

(c) We consider now graphs with a purely bosonic super-loop. One obtains the following type of distribution:

$$d_{(123)}(x_1, x_2; x_3) = \Delta_3^{AF}(x_1 - x_2)\big[\Delta_1^{(+)}(x_2 - x_3)\Delta_2^{(-)}(x_3 - x_1)$$
$$-\Delta_1^{(-)}(x_2 - x_3)\Delta_2^{(+)}(x_3 - x_1)\big]$$
$$+\Delta_2^F(x_3 - x_1)\big[\Delta_3^{(+)}(x_1 - x_2)\Delta_1^{(-)}(x_2 - x_3)$$
$$-\Delta_3^{(-)}(x_1 - x_2)\Delta_1^{(+)}(x_2 - x_3)\big]$$
$$+\Delta_1^F(x_2 - x_3)\big[\Delta_1^{(+)}(x_3 - x_1)\Delta_3^{(-)}(x_1 - x_2)$$
$$-\Delta_1^{(-)}(x_3 - x_1)\Delta_3^{(+)}(x_1 - x_2)\big]; \quad (5.1.21)$$

for a fermionic super-loop an overall -1 sign appears.

The causal support property can be checked by deriving two alternative formulæ:

$$\begin{aligned}
d_{(123)}(x_1,x_2;x_3) = &-\Delta_3^{\rm ret}(x_1-x_2)\big[\Delta_1^{(+)}(x_2-x_3)\Delta_2^{(-)}(x_3-x_1)\\
&-\Delta_1^{(-)}(x_2-x_3)\Delta_2^{(+)}(x_3-x_1)\big]\\
&+\Delta_2^{\rm adv}(x_3-x_1)\big[\Delta_3^{(+)}(x_1-x_2)\Delta_1^{(-)}(x_2-x_3)\\
&-\Delta_3^{(-)}(x_1-x_2)\Delta_1^{(+)}(x_2-x_3)\big]\\
&+\Delta_1^{\rm adv}(x_2-x_3)\big[\Delta_1^{(+)}(x_3-x_1)\Delta_3^{(-)}(x_1-x_2)\\
&-\Delta_1^{(-)}(x_3-x_1)\Delta_3^{(+)}(x_1-x_2)\big]\\
= &-\Delta_3^{\rm adv}(x_1-x_2)\big[\Delta_1^{(+)}(x_2-x_3)\Delta_2^{(-)}(x_3-x_1)\\
&-\Delta_1^{(-)}(x_2-x_3)\Delta_2^{(+)}(x_3-x_1)\big]\\
&+\Delta_2^{\rm ret}(x_3-x_1)\big[\Delta_3^{(+)}(x_1-x_2)\Delta_1^{(-)}(x_2-x_3)\\
&-\Delta_3^{(-)}(x_1-x_2)\Delta_1^{(+)}(x_2-x_3)\big]\\
&+\Delta_1^{\rm ret}(x_2-x_3)\big[\Delta_1^{(+)}(x_3-x_1)\Delta_3^{(-)}(x_1-x_2)\\
&-\Delta_1^{(-)}(x_3-x_1)\Delta_3^{(+)}(x_1-x_2)\big].
\end{aligned}\tag{5.1.22}$$

We denote suggestively this type of distribution by $d_{(123)}(\Delta_1,\Delta_2,\Delta_3)(x_1,x_2;x_3)$ where Δ_i, $i=1,2,3$ are distributions from the lists (4.1.9)–(4.1.12) and we have concerning the order of singularity

$$\omega(d_{(123)}(\Delta_1,\Delta_2,\Delta_3)) = 4+\sum_i \omega(\Delta_i). \tag{5.1.23}$$

We say now something about the generic momentum space structure of such a distribution. First one has to obtain from the explicit formulæ for the distributions Δ in one variable that in all cases

$$\tilde{\Delta}_i^{(\pm)}(p) \sim \theta(\pm p_0) f_i(p^2) \tag{5.1.24}$$

with $\text{supp}(f_i)\subset\{p^2\geqslant\lambda_i^2\}$ for some parameters with mass significance $\lambda_i\geqslant 0$, $i=1,2,3$. We consider now the Taylor transform of $\Delta_{(123)}(\xi_1,\xi_2)$ and we use the notation $K\equiv k_1+k_2$; the generic structure is

$$\tilde{\Delta}_{(123)}(k_1,k_2) = \theta(k_1^2-(\lambda_2+\lambda_3)^2)g_1+\theta(k_2^2-(\lambda_3+\lambda_1)^2)g_2+\theta(K^2-(\lambda_1+\lambda_2)^2)g_3. \tag{5.1.25}$$

It follows that if at least two of the masses $\lambda_i\geqslant 0$, $i=1,2,3$ are strictly positive, then $(0,0)\notin\text{supp}(\tilde{\Delta}_{(123)}(k_1,k_2)$. This observation is useful because for causal distributions with such support property in momentum space one can use the so-called *central* formula for causal decomposition of distributions [23]. If the conditions of validity of the central formula are not met we will have to use a regularization procedure.

(ii) We investigate the possible Ward identities and obstructions to causal splitting. First we consider case (b). We illustrate this case on the the distribution

$$d^\mu \equiv d_{(3)}(\partial^\mu D_m,\Delta) \tag{5.1.26}$$

where Δ is arbitrary. The other cases can be treated similarly. First we derive the Ward identity

$$\partial_\mu d^\mu = -\delta(x_1-x_3)\Delta(x_2-x_3)+m^2D_m^F(x_1-x_3)\Delta(x_2-x_3). \tag{5.1.27}$$

Using the formula for the causal splitting (5.1.13) one can see that the preceding identity is preserved by the operation of distribution splitting.

Next, we consider case (c). We have to study separately the case when the super-loop contains at most one Dirac line and the case when we have three Dirac lines. We illustrate the first case by the distribution

$$d^\mu = d_{(123)}(\partial^\mu D_{m_1}, D_{m_2}, D_{m_3}) \qquad m_1 > 0; \tag{5.1.28}$$

the other cases can be treated similarly. The Ward identity is in this case

$$\partial_\mu d^\mu = -\delta(x_2 - x_3) D_{m_2,m_3}(x_3 - x_1) + \cdots \tag{5.1.29}$$

where by ... we mean contributions with the order of singularity strictly smaller than zero. One computes immediately that both sides have the order of singularity equal to unity. If we have $m_2+m_3 > 0$ then we can apply the central decomposition formula and obtain no anomaly. In the opposite case, we use the standard regularization procedure (5.1.1) of the distributions appearing in the lists (4.1.9)–(4.1.12) presented at the beginning of this subsection. The decomposition (5.1.1) induces a similar decomposition for the distributions of the type $d_{(i)}$:

$$d_{(i)} = d_{(i)}^0 + d^{\rm reg} \tag{5.1.30}$$

where

$$\omega(d^0) = \omega(d) \qquad \omega(d^{\rm reg}) = \omega(d) - 2 \tag{5.1.31}$$

and the support properties of d^0 in the momentum space are more convenient: $(0, 0) \notin \mathrm{supp}(\tilde{d}_{(i)}^0)$.

If we apply this decomposition to the distributions d^μ and d we obtain two Ward identities, one for each part. The first one can be split causally without anomalies using the central decomposition formula. For the second identity we note that both sides have order of singularity strictly lower than -1 so this relation can be also split causally without anomalies as explained at the end of section 3. In this way we can obtain a anomaly-free decomposition of the Ward identity we have started with. One has to check case by case this argument for all the other types of distribution of type (c) without Dirac loops.

A very important observation is that the preceding argument is not valid for distributions associated with super-loops containing three Dirac lines. The reason is that one is led to the computation of some traces. To be more specific the relevant terms from the first commutator (5.1.9) are

$$\begin{aligned} D_1^\mu(x_1, x_2; x_3) = {} & d_{abc}^{\mu\nu\rho}(x_1, x_2; x_3) : u_a(x_1) A_{b\nu}(x_2) A_{c\rho}(x_3) : \\ & + d_{abc}^{\mu\nu}(x_1, x_2; x_3) : u_a(x_1) \Phi_b(x_2) A_{c\nu}(x_3) : + (x_2 \leftrightarrow x_3) \\ & + d_{abc}^{\mu}(x_1, x_2; x_3) : u_a(x_1) \Phi_b(x_2) \Phi_c(x_3) : \\ & + d_a^\mu(x_1, x_2; x_3) u_a(x_1) + \cdots \end{aligned} \tag{5.1.32}$$

where by $\cdots$ we mean the terms which cannot produce anomalies. Let us note that all these terms are obtained from Wick contractions of the pieces of the interaction Lagrangian of the type (2.2.7).

The distributions appearing in this formula are sums of distributions of the type $d_{(123)}$ because of the traces, but in this case the trace operation can annihilate the most singular term and instead of (5.1.23) we might have

$$\omega(d) < 4 + \sum_i \omega(\Delta_i). \tag{5.1.33}$$

It follows that these distributions must be studied separately and some explicit computation are required.

(iii) All the distributions appearing in the formula (5.1.32) have eight contributions corresponding to the decomposition of the three currents involved in (2.2.7) into the vectorial and axial components. If we compute the contribution corresponding to three vectorial factors, then the others can be obtained by simple substitutions. Let us consider this pure vector contribution to $d^{\mu\nu\rho}_{abc}(x_1, x_2; x_3)$; the explicit expression is

$$d^{\mu\nu\rho}_{abc;VVV} = (t_a)_{AB}(t_b)_{BC}(t_c)_{CA} d^{\mu\nu\rho(V)}_{M_C,M_A,M_B} + (t_a)_{AB}(t_c)_{BC}(t_b)_{CA} d^{\nu\mu\rho(V)}_{M_C,M_B,M_A} \tag{5.1.34}$$

where we have defined for arbitrary masses M_1, M_2, M_3 the following fundamental distribution

$$\begin{aligned} d^{\mu\nu\rho(V)}_{M_1,M_2,M_3}(x_1, x_2; x_3) = \mathrm{Tr}\,\bigl\{ & S^{AF}_{M_3}(x_1 - x_2)\gamma^\nu \bigl[S^{(-)}_{M_1}(x_2 - x_3)\gamma^\rho S^{(+)}_{M_2}(x_3 - x_1) \\ & - S^{(+)}_{M_1}(x_2 - x_3)\gamma^\rho S^{(-)}_{M_2}(x_3 - x_1)\bigr]\gamma^\mu \\ & + S^F_{M_1}(x_2 - x_3)\gamma^\rho \bigl[S^{(-)}_{M_2}(x_3 - x_1)\gamma^\mu S^{(+)}_{M_3}(x_1 - x_2) \\ & - S^{(+)}_{M_2}(x_3 - x_1)\gamma^\rho S^{(-)}_{M_3}(x_1 - x_2)\bigr]\gamma^\nu \\ & + S^F_{M_2}(x_3 - x_1)\gamma^\mu \bigl[S^{(-)}_{M_3}(x_1 - x_2)\gamma^\nu S^{(+)}_{M_1}(x_2 - x_3) \\ & - S^{(+)}_{M_3}(x_1 - x_2)\gamma^\nu S^{(-)}_{M_1}(x_2 - x_3)\bigr]\gamma^\rho \bigr\} \end{aligned} \tag{5.1.35}$$

which is similar to (5.1.21); compare also to formula (5.3.11) from [23]. It also has causal support: one can obtain quite easily alternative expressions having the structure (5.1.22).

The entire vectorial contribution is now obtained if we add the contributions following from $d^{\mu\nu\rho}_{abc;VVV}$ if we perform the following simple transforms:

$$t_a \to t'_a \qquad t_b \to t'_b \qquad \gamma^\mu \to \gamma^\mu\gamma_5 \qquad \gamma^\nu \to \gamma^\nu\gamma_5 \tag{5.1.36}$$

and the two other similar possibilities. Using the formula (4.1.13) we obtain the following form for the pure vectorial part:

$$\begin{aligned} d^{\mu\nu\rho}_{abc;V} = {} & (t_a)_{AB}(t_b)_{BC}(t_c)_{CA} d^{\mu\nu\rho(V)}_{M_C,M_A,M_B} + (t_a)_{AB}(t_c)_{BC}(t_b)_{CA} d^{\nu\mu\rho(V)}_{M_C,M_B,M_A} \\ & + (t'_a)_{AB}(t'_b)_{BC}(t_c)_{CA} d^{\mu\nu\rho(V)}_{M_C,M_A,-M_B} + (t'_a)_{AB}(t_c)_{BC}(t'_b)_{CA} d^{\nu\mu\rho(V)}_{M_C,M_B,-M_A} \\ & + (t_a)_{AB}(t'_b)_{BC}(t'_c)_{CA} d^{\mu\nu\rho(V)}_{-M_C,M_A,M_B} + (t_a)_{AB}(t'_c)_{BC}(t'_b)_{CA} d^{\nu\mu\rho(V)}_{-M_C,M_B,M_A} \\ & + (t'_a)_{AB}(t_b)_{BC}(t'_c)_{CA} d^{\mu\nu\rho(V)}_{M_C,-M_A,M_B} + (t'_a)_{AB}(t'_c)_{BC}(t_b)_{CA} d^{\nu\mu\rho(V)}_{M_C,-M_B,M_A}. \end{aligned} \tag{5.1.37}$$

One notices that the vectorial part of $d^{\mu\nu\rho}_{abc}$ is expressed only in terms of the distribution of the type $d^{\mu\nu\rho(V)}_{M_1,M_2,M_3}$.

By similar transforms one can obtain the pure axial part. One defines in analogy to (5.1.34) the distribution $d^{\mu\nu\rho(A)}_{M_1,M_2,M_3}$ by inserting a factor γ_5:

$$d^{\mu\nu\rho(A)}_{M_1,M_2,M_3}(x_1, x_2; x_3) = \mathrm{Tr}\,\gamma_5\,\{\cdots\} \tag{5.1.38}$$

where by $\{\cdots\}$ we mean the same parenthesis as in (5.1.35). The pure axial contribution to $d^{\mu\nu\rho}_{abc}$ is similar to (5.1.37). The only relevant thing is that it is expressed only in terms of the new distribution $d^{\mu\nu\rho(A)}_{M_1,M_2,M_3}$. It follows therefore that the distribution $d^{\mu\nu\rho}_{abc}$ can be expressed in terms of two independent distributions: $d^{\mu\nu\rho(V)}_{M_1,M_2,M_3}$ and $d^{\mu\nu\rho(A)}_{M_1,M_2,M_3}$. One can prove quite easily that the orders of singularities are

$$\omega(d^{\mu\nu\rho(V)(A)}_{M_1,M_2,M_3}) = 1. \tag{5.1.39}$$

Let us note in passing that the asymptotic behaviour of the distribution

$$d^{\mu\nu\rho}_{abc} = d^{\mu\nu\rho}_{abc;V} + d^{\mu\nu\rho}_{abc;A} \tag{5.1.40}$$

is given by

$$d^{\mu\nu\rho}_{abc} \sim V_{abc} d^{\mu\nu\rho}_{(V)} + A_{abc} d^{\mu\nu\rho}_{(A)} \tag{5.1.41}$$

where the axial tensor A_{abc} is given by the expression (1.0.1) from the introduction, the vector tensor is given by a similar expression

$$V_{abc} \equiv \operatorname{Tr}\left(t_a^+[t_b^+, t_c^+] + t_a^-[t_b^-, t_c^-]\right) = f_{bcd} \operatorname{Tr}\left(t_a^+ t_d^+ + t_a^- t_d^-\right) \tag{5.1.42}$$

and

$$d^{\mu\nu\rho}_{(V)(A)} \equiv d^{\mu\nu\rho(V)(A)}_{0,0,0}. \tag{5.1.43}$$

A similar investigation can be performed for the other distributions appearing in the formula (5.1.32). The distribution $d^{\mu\nu}_{abc}$ can be expressed in terms of two independent distributions: $d^{\mu\nu(V)}_{M_1,M_2,M_3}$ and $d^{\mu\nu(A)}_{M_1,M_2,M_3}$, which can be obtained from $d^{\mu\nu\rho(V)}_{M_1,M_2,M_3}$ and $d^{\mu\nu\rho(A)}_{M_1,M_2,M_3}$ making $\gamma^\rho \to 1$. The order of singularity of these distributions is lower than naive power counting indicates. They can be written as follows:

$$d^{\mu\nu(V)(A)}_{M_1,M_2,M_3} = \sum_{i=1}^{3} M_i d_i^{\mu\nu(V)(A)} \tag{5.1.44}$$

with

$$\omega(d_i^{\mu\nu(V)(A)}) = 0. \tag{5.1.45}$$

Analogously, the distribution d^{μ}_{abc} can be expressed in terms of two independent distributions: $d^{\mu(V)}_{M_1,M_2,M_3}$ and $d^{\mu(A)}_{M_1,M_2,M_3}$, which can be obtained from $d^{\mu\nu(V)}_{M_1,M_2,M_3}$ and $d^{\mu\nu(A)}_{M_1,M_2,M_3}$ making $\gamma^\nu \to 1$. We also have

$$\omega(d^{\mu(V)(A)}_{M_1,M_2,M_3}) = 1. \tag{5.1.46}$$

Finally, the distribution d^{μ}_{a} can be expressed in terms of two independent distributions: $d^{\mu(V)}_{m;M_1,M_2,M_3}$ and $d^{\mu(A)}_{m;M_1,M_2,M_3}$, which can be obtained from $d^{\mu(V)}_{M_1,M_2,M_3}$ and $d^{\mu(A)}_{M_1,M_2,M_3}$ by making $S_{M_1} \to \Sigma_{m,M_1}$. We have in this case

$$\omega(d^{\mu(V)(A)}_{m;M_1,M_2,M_3}) = 3. \tag{5.1.47}$$

We also need the distributions $d^{(V)}_{M_1,M_2,M_3}$ and $d^{(V)}_{m;M_1,M_2,M_3}$, which can be obtained from $d^{\mu(V)}_{M_1,M_2,M_3}$ and $d^{\mu(V)}_{m;M_1,M_2,M_3}$ respectively by making $\gamma^\mu \to 1$. In this case we have a structure similar to (5.1.44) and a similar result for the order of singularity.

We can easily see that all distributions $d^{\cdots(A)}_{\cdots}$ are completely antisymmetric in the Lorentz indices due to traces involving a γ_5 matrix. It is not difficult to prove that one can impose a supplementary condition on the causal splitting procedure, namely the preservation of this symmetry property.

The distributions appearing in the third commutator from (5.1.9) can be obtained from the preceding ones by making the substitution $x_1 \leftrightarrow x_2$ and this doubles the value of the possible anomalies originating from the first commutator.

The distributions appearing in the second commutator from (5.1.9) can be obtained from the preceding ones by more subtle transforms. For case (b) and case (c) without Dirac loops we have the same list of distributions and there are no anomalies. For case (c) with Dirac loops we have to consider

$$\begin{aligned}
&d^{\mu\nu\rho}_{M_1,M_2,M_3} \to d^{\rho\nu\mu}_{M_1,M_2,M_3}\\
&d^{\mu\nu}_{M_1,M_2,M_3}(x_1,x_2;x_3) \to f^{\mu\nu}_{M_2,M_3,M_1}(x_1,x_2;x_3) \equiv d^{\nu\mu}_{M_2,M_3,M_1}(x_2,x_3;x_1)\\
&d^{\mu}_{M_1,M_2,M_3}(x_1,x_2;x_3) \to f^{\mu}_{M_1,M_2,M_3}(x_1,x_2;x_3) \equiv d^{\mu}_{M_2,M_3,M_1}(x_3,x_1;x_2)\\
&d^{\mu}_{m;M_1,M_2,M_3}(x_1,x_2;x_3) \to f^{\mu}_{m;M_1,M_2,M_3}(x_1,x_2;x_3) \equiv d^{\mu}_{m;M_2,M_3,M_1}(x_3,x_1;x_2).
\end{aligned} \tag{5.1.48}$$

(iv) Now we can give the list of Ward identities verified by these distributions. Using the Dirac equation for the propagators we obtain

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial x_1^\mu} d^{\mu\nu\rho(V)}_{M_1,M_2,M_3}(x_1,x_2;x_3) &= (M_2-M_3) f^{\nu\rho(V)}_{M_1,M_2,M_3}(x_1,x_2;x_3) \\ &\quad +\mathrm{i}\delta(x_1-x_2)P^{\nu\rho}_{M_1,M_2}(x_1-x_3)+\mathrm{i}\delta(x_1-x_3)P^{\rho\nu}_{M_1,M_3}(x_2-x_3) \end{aligned} \tag{5.1.49}$$

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial x_3^\mu} d^{\rho\nu\mu(V)}_{M_1,M_2,M_3}(x_1,x_2;x_3) &= (M_2-M_1) d^{\rho\nu(V)}_{M_1,M_2,M_3}(x_1,x_2;x_3) \\ &\quad +\mathrm{i}\delta(x_1-x_3)P^{\rho\nu}_{M_1,M_3}(x_2-x_3)-\mathrm{i}\delta(x_2-x_3)P^{\rho\nu}_{M_2,M_3}(x_1-x_3) \end{aligned} \tag{5.1.50}$$

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial x_1^\mu} d^{\mu\nu(V)}_{M_1,M_2,M_3}(x_1,x_2;x_3) &= (M_2-M_3) d^{\nu(V)}_{M_2,M_3,M_1}(x_2,x_3;x_1) \\ &\quad +\mathrm{i}\delta(x_1-x_2)P^{\nu}_{M_1,M_2}(x_1-x_3)+\mathrm{i}\delta(x_1-x_3)P^{\nu}_{M_1,M_3}(x_2-x_3) \end{aligned} \tag{5.1.51}$$

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial x_3^\mu} f^{\mu\nu(V)}_{M_1,M_2,M_3}(x_1,x_2;x_3) &= (M_2-M_1) d^{\nu(V)}_{M_1,M_2,M_3}(x_2,x_3;x_1) \\ &\quad +\mathrm{i}\delta(x_1-x_3)P^{\nu}_{M_1,M_3}(x_2-x_3)-\mathrm{i}\delta(x_2-x_3)P^{\nu}_{M_2,M_3}(x_1-x_3) \end{aligned} \tag{5.1.52}$$

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial x_1^\mu} d^{\mu(V)}_{M_1,M_2,M_3}(x_1,x_2;x_3) &= (M_2-M_3) d^{(V)}_{M_1,M_2,M_3}(x_1,x_2;x_3) \\ &\quad +\mathrm{i}\delta(x_1-x_3)P_{M_1,M_2}(x_2-x_3)+\mathrm{i}\delta(x_2-x_3)P_{M_1,M_3}(x_1-x_3) \end{aligned} \tag{5.1.53}$$

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial x_1^\mu} d^{\mu(V)}_{m;M_1,M_2,M_3}(x_1,x_2;x_3) &= (M_2-M_3) d^{(V)}_{m;M_1,M_2,M_3}(x_1,x_2;x_3) \\ &\quad +\mathrm{i}\delta(x_1-x_3)P_{m;M_1,M_2}(x_2-x_3)+\mathrm{i}\delta(x_2-x_3)P_{m;M_1,M_3}(x_1-x_3). \end{aligned} \tag{5.1.54}$$

The Ward identities for the axial distributions present a notable difference. Because of the trace operation, the delta terms disappear. Using also formula (4.1.13) we obtain

$$\mathrm{i}\frac{\partial}{\partial x_1^\mu} d^{\mu\nu\rho(A)}_{M_1,M_2,M_3}(x_1,x_2;x_3) = (M_2+M_3) f^{\nu\rho(A)}_{M_1,M_2,M_3}(x_1,x_2;x_3) \tag{5.1.55}$$

$$\mathrm{i}\frac{\partial}{\partial x_3^\mu} d^{\rho\nu\mu(A)}_{M_1,M_2,M_3}(x_1,x_2;x_3) = (M_1+M_2) d^{\nu\rho(A)}_{M_1,M_2,M_3}(x_1,x_2;x_3) \tag{5.1.56}$$

$$\mathrm{i}\frac{\partial}{\partial x_1^\mu} d^{\mu\nu(A)}_{M_1,M_2,M_3}(x_1,x_2;x_3) = (M_2+M_3) d^{\nu(A)}_{M_1,M_2,M_3}(x_2,x_3;x_1) \tag{5.1.57}$$

$$\mathrm{i}\frac{\partial}{\partial x_3^\mu} f^{\mu\nu(A)}_{M_1,M_2,M_3}(x_1,x_2;x_3) = (M_1+M_2) d^{\nu(A)}_{M_1,M_2,M_3}(x_2,x_3;x_1) \tag{5.1.58}$$

$$\mathrm{i}\frac{\partial}{\partial x_1^\mu} d^{\mu(A)}_{M_1,M_2,M_3}(x_1,x_2;x_3) = 0 \tag{5.1.59}$$

$$\mathrm{i}\frac{\partial}{\partial x_1^\mu} d^{\mu(A)}_{m;M_1,M_2,M_3}(x_1,x_2;x_3) = 0. \tag{5.1.60}$$

The causal splitting of these two types of Ward identity is sensibly different. Let us first consider only the first six equations (the vectorial Ward identities). Because of the delta terms in the right-hand sides, we have the same order of singularity for both sides in all vectorial Ward identities; if the conditions of application of the central splitting formula are met we obtain no anomalies. If some of the masses are null, one has to use a regularization procedure as for case (b). More precisely one can prove that the decomposition (5.1.1) induces a similar decomposition for the distributions of the type $d_{(123)}$:

$$d_{(123)} = d^0_{(123)} + d^{\mathrm{reg}}_{(123)} \tag{5.1.61}$$

where

$$\omega(d^0_{(123)}) = \omega(d_{(123)}) \qquad \omega(d^{\text{reg}}_{(123)}) = \omega(d_{(123)}) - 4 \tag{5.1.62}$$

and the support properties of $d^0_{(123)}$ in the momentum space are more convenient: $(0, 0) \notin \text{supp}(\tilde{d}^0_{(123)})$.

We turn now to the last six equations (the axial Ward identities). The previous argument is still valid for the last two of them. We consider some generic anomalies P_1^ν, P_3^ν obtained after the causal splitting of the identities (5.1.57) and (5.1.58). If we differentiate the corresponding equations with respect to x_3^ν and x_1^ν respectively, we obtain from antisymmetry the consistency equations

$$\frac{\partial}{\partial x_3^\nu} P_1^\nu = 0 \qquad \frac{\partial}{\partial x_1^\nu} P_3^\nu = 0 \tag{5.1.63}$$

and this leads to $P_i^\mu = 0, i = 1, 3$.

The Ward identities (5.1.55) and (5.1.56) can produce anomalies of the type

$$P^{\nu\rho}(X) = \text{const} \times \varepsilon^{\nu\rho\alpha\beta} \frac{\partial^2}{\partial x_1^\alpha \partial x_2^\beta} \delta(X) \tag{5.1.64}$$

for some positive constant const. The explicit expression of this constant can be computed as in [23] section 5.3. One cannot eliminate such a type of anomaly from both equations by redefinition. The resulting anomaly is then

$$A(X) = \text{const} \times A_{abc} \varepsilon^{\nu\rho\alpha\beta} \frac{\partial^2}{\partial x_1^\alpha \partial x_2^\beta} \delta(X) : u_a(x_1) A_{b\nu}(x_2) A_{c\rho}(x_3) : \tag{5.1.65}$$

where

$$A_{abc} \equiv 2\, \text{Tr} \left(t'_a \{t'_b, t'_c\} + t_a \{t_b, t'_c\} + t'_a \{t_b, t_c\} + t_a \{t'_b, t'_c\} \right). \tag{5.1.66}$$

Performing some redefinitions of the expressions $A_l^\mu(X)$ ('integration by parts') we can reexpress this *axial anomaly* in the following form:

$$A_{ABBJ}(X) = \text{const} \times A_{abc} \varepsilon_{\mu\nu\rho\sigma} \delta(X) : u_a(x_1) F_b^{\mu\nu}(x_1) F_c^{\rho\sigma}(x_3) : \tag{5.1.67}$$

and we can also show that the tensor depending only on the group indices A_{abc} is in fact given by the formula (1.0.1) from the introduction. The anomaly A_{ABBJ} is a cocycle

$$d_Q A_{ABBJ} = 0 \tag{5.1.68}$$

but it is not a coboundary, so disappears *iff* we have the condition $A_{abc} = 0$ i.e. the well known condition (5.1.6) from the statement.

(v) We still have to investigate the possible anomalies originating from the delta terms, i.e. from distributions associated with graphs of type (a). We present here briefly the analysis of these terms. One can compute the commutators and select the terms which will lead, in principle, to an anomaly. We obtain

$$\begin{aligned}
[T_1^\mu(x), L(y)] = {} & f_{abc} f_{dcf} f_{dgh} \frac{\partial}{\partial x_\mu} D_{m_c}(x-y) : u_a(x) A_{b\nu}(x) A_{f\lambda}(y) A_g^\nu(y) A_h^\lambda(y) : \\
& -2 f_{abc} f'_{dec} f'_{dgh} \frac{\partial}{\partial x_\mu} D_{m_c}(x-y) : u_a(x) A_{b\rho}(x) A_h^\rho(y) \Phi_e(y) \Phi_g(y) : \\
& +2 f'_{abc} f'_{dbf} f'_{dgh} \frac{\partial}{\partial x_\mu} D_{m_b^*}(x-y) : \Phi_a(x) u_c(x) A_{f\rho}(y) A_h^\rho(y) \Phi_g(y) : \\
& +4 f'_{abc} g'_{bfgh} \frac{\partial}{\partial x_\mu} D_{m_b^*}(x-y) : \Phi_a(x) u_c(x) \Phi_f(y) \Phi_g(y) \Phi_h(y) : + \cdots .
\end{aligned} \tag{5.1.69}$$

By $\cdots$ we mean the rest of the commutator which cannot produce anomalies. Now, as in [16] and [17] we obtain from this commutator a possible anomaly:

$$A(x_1, x_2, x_3) = \delta(X)\big[A_1(x_1) + A_2(x_1) + A_3(x_1) + A_4(x_1)\big] \tag{5.1.70}$$

where

$$A_1(x) = \mathrm{i} f_{abc} f_{dcf} f_{dgh} : u_a(x) A_{b\nu}(x) A_{f\lambda}(x) A_g^{\nu}(x) A_h^{\lambda}(x) : \tag{5.1.71}$$

$$A_2(x) = -2\mathrm{i} f_{abc} f'_{dec} f'_{dgh} : u_a(x) A_{b\rho}(x) A_h^{\rho}(x) \Phi_e(x) \Phi_g(x) : \tag{5.1.72}$$

$$A_3(x) = 2\mathrm{i} f'_{abc} f'_{dbf} f'_{dgh} : \Phi_a(x) u_c(x) A_{f\rho}(x) A_h^{\rho}(x) \Phi_g(x) : \tag{5.1.73}$$

$$A_4(x) = \mathrm{i}\big[f'_{abc} g'_{bfgh} + f'_{fbc} g'_{bagh} + f'_{gbc} g'_{bafh} + f'_{hbc} g'_{bafg}\big] : \\ \times \Phi_a(x) u_c(x) \Phi_f(y) \Phi_g(x) \Phi_h(x) : . \tag{5.1.74}$$

The results are:

- In [11] it is proved that $A_1 = 0$ due to the Jacobi identity.
- One can also show, using the identity (2.1.19), that $A_2 + A_3 = 0$.
- If we try to write the anomaly A_4 as a coboundary $d_Q L(x)$ we should take

$$L(x) = g'_{acfgh} : \Phi_a(x) \Phi_c(x) \Phi_f(x) \Phi_g(x) \Phi_h(x) : \tag{5.1.75}$$

which is forbidden by the assumption that (3.1.10) is fulfilled.

So we obtain the second restriction from the statement. □

Remark 5.2. Recently [19] a new method was proposed to solve problems of consistency such as those appearing in our paper. Instead of imposing a factorization condition of the type (2.1.10) (or its 'infinitesimal' version (2.1.11)) one imposes a quantum analogue of the Noether conservation law of a certain current. Presumably, this starting points are equivalent and they should lead to the same sets of consistency conditions. This point deserves further investigation. However, one should compare carefully the relation (4.2.8) expressing the conservation law of the BRST current (and equivalent to the formal adiabatic limit condition) to the relation (4.5) of [19] expression the quantum Noether postulate.

5.2. The standard model

We recall the notations from [17]. The Lie algebra is in this case $su(2) \times u(1)$ and the standard basis X_a, $a = 0, 1, 2, 3$, has the usual commutation relations

$$[X_a, X_b] = \epsilon_{abc} X_c \qquad a, b = 1, 2, 3 \qquad [X_0, X_a] = 0 \quad a = 1, 2, 3. \tag{5.2.1}$$

In the new basis Y_a, $a = 0, 1, 2, 3$, defined by

$$Y_a = g\ X_a \quad a = 1, 2 \qquad Y_3 = -g\ \cos\theta\ X_3 + g'\ \sin\ \theta X_0 \\ Y_0 = -g\ \sin\theta\ X_3 - g'\ \cos\ \theta X_0 \tag{5.2.2}$$

(here the angle θ, determined by the condition $\cos\ \theta > 0$ is the *Weinberg angle* and the constants g and g' are real with $g > 0$) the structure constants are

$$f_{210} = g\ \sin\ \theta \qquad f_{321} = g\ \cos\ \theta \qquad f_{310} = 0 \qquad f_{320} = 0 \tag{5.2.3}$$

and the rest of the constants are determined by antisymmetry. The choice of the masses is

$$m_0 = 0 \qquad m_a \neq 0 \qquad a = 1, 2, 3 \tag{5.2.4}$$

(the particles created by A_0^{μ} being the *photons* and the particles created by A_a^{μ}, $a = 1, 2, 3$, the *heavy bosons*).

In [17] we found the following result.

Theorem 5.3. *In the SM, the following relations are true:*

(a) The masses of the heavy bosons are constrained by

$$m_1 = m_2 = m_3 \cos\,\theta. \tag{5.2.5}$$

(b) The constants f'_{abc} are completely determined by the antisymmetry property (2.1.17) and

$$\begin{array}{lll} f'_{011} = f'_{022} = \dfrac{\epsilon\, g}{2} & f'_{033} = \dfrac{\epsilon\, g}{2\cos\,\theta} & f'_{210} = g\ \sin\,\theta \\ f'_{321} = -f'_{312} = \dfrac{g}{2} & f'_{123} = -g\ \dfrac{\cos\,2\theta}{2\cos\,\theta} & \end{array} \tag{5.2.6}$$

the rest of them being zero. Here ϵ can take the values + or −.

(c) The constants f''_{abc} are (partially) determined by

$$\begin{array}{ll} f''_{abc} = 0 \quad (a,b,c = 1,2,3) & f''_{001} = f''_{002} = f''_{003} = f''_{012} = f''_{023} = f''_{031} = 0 \\ f''_{011} = f''_{022} = f''_{033} = \dfrac{\epsilon g}{12 m_1}\left(m_0^H\right)^2. & \end{array} \tag{5.2.7}$$

Moreover, one can fix $\epsilon = +$.

Remark 5.4. In [24] a dual point of view is followed: one gives the masses of the heavy bosons $m_1 = m_2 \neq m_3$ and determines that the gauge algebra must be $su(2) \times u(1)$.

We consider the minimal SM containing only one generation of Dirac particles. In this case one takes in the generic formalism from the preceding section $N = 2$ and

$$M \equiv \begin{pmatrix} 0 & 0 \\ 0 & m_{\mathrm{e}} \end{pmatrix}. \tag{5.2.8}$$

The components ψ_2 (ψ_1) correspond to the *electron* (the electronic *neutrino*) and m_{e} is the electron mass. Remark that the neutrino mass is considered null.

The choice for the representations $t_a^{\pm}$ is the following one:

$$\begin{array}{ll} t_1^+ = \frac{1}{2} g \sigma_1 & t_3^+ = \frac{1}{2}\big(-g\cos\,\theta\sigma_3 + g'\sin\,\theta\mathbf{1}\big) \\ t_2^+ = \frac{1}{2} g \sigma_2 & t_0^+ = -\frac{1}{2}\big(g\sin\,\theta\sigma_3 + g'\cos\,\theta\mathbf{1}\big) \end{array} \tag{5.2.9}$$

and

$$t_1^+ = t_2^+ = 0 \qquad t_3^+ = y\ \sin\,\theta \qquad t_0^+ = -y\ \cos\,\theta \tag{5.2.10}$$

where σ_i are the Pauli matrices. The representation property (4.1.5) is fulfilled for any matrix y. However, we have the following elementary result.

Proposition 5.5. *The interaction between the Dirac field of the electron ψ_2 and the electromagnetic field A_0^{μ} has the usual form*

$$e : \bar{\psi}_2 \gamma_\mu \psi_2 : A_0^{\mu}$$

(here e is the electron charge) iff

$$g = \frac{e}{\sin\,\theta} \qquad g' = -\frac{e}{\cos\,\theta} \tag{5.2.11}$$

and

$$y = \frac{1}{2} g'(\mathbf{1} - \sigma_3) = \frac{1}{2m_{\mathrm{e}}} g' M. \tag{5.2.12}$$

Next, we have:

Proposition 5.6. *The expressions for the matrices* s_a^+ *are*

$$\begin{aligned} s_0^+ &= \frac{m_\mathrm{e}}{2m_1}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \qquad & s_1^+ &= \frac{\mathrm{i}m_\mathrm{e}}{2m_1}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\\ s_2^+ &= -\frac{m_\mathrm{e}}{2m_1}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \qquad & s_3^+ &= \frac{\mathrm{i}m_\mathrm{e}}{2m_1}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \end{aligned} \tag{5.2.13}$$

Proof. One uses the relations (2.2.18) for $a = 1, 2, 3$ and obtains the expressions for s_a^+, $a = 1, 2, 3$. Next, we use the relation (4.1.6), more precisely

$$t_a^- s_0^+ - s_0^+ t_a^+ = \mathrm{i} f'_{0ca} s_c^+ \quad a = 1, 2, 3. \tag{5.2.14}$$

This equation gives immediately the expression for s_0^+. □

The expression of the Higgs potential is obtained as in [4, 13]. One can check that in this way the usual SM is obtained.

5.3. Regularization and anomalies

We have succeeded in giving a complete analysis of the possible anomalies appearing in the SM up to the third order of the perturbation theory. One would want to generalize this analysis to all orders of the perturbation theory. It is possible that one can use the same type of combinatorial argument, namely one considers possible distributions appearing in the commutators $D(X)$ of order n and observes that only the super-loop graphs with Dirac lines can produce anomalies. Then it is quite possible that in higher orders the orders of singularity are sufficiently lower to make possible a causal splitting of the Ward identities without anomalies. This seems to be indicated by the traditional argument from the literature [1, 2, 25].

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